Notes from the Margin

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Tracking the Multiplicity of Zero Using Maximum Matchings

By: Johnna Parenteau (University of Regina)

Johnna Parenteau

Mathematics isn't just for the smart and the talented. It's for the curious and passionate individuals who aim to understand the world and empower others to think logically, creatively, and abstractly.

Matchings have a rich history in graph theory due to their vast applications in network and scheduling problems, theoretical chemistry, and even statistical physics. Known as the reference, acyclic, or king polynomial, depending on the field of interest, the matching polynomial has had many differing notations throughout the years; the most famous is the matching polynomial as defined in [1]. A limitation of this polynomial is that it only correlates to unweighted graphs, but we can extend it by redefining the polynomial as follows. Suppose $G = (V, E, w)$ is a graph with a fixed weight function, w, and $n = |V(G)|$. Let

$$m_w(G,x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \mu_w(G,k) x^{n-2k}$$

where

$$\mu_w(G,k) = \sum_{k-\text{matchings in } G} \quad \prod_{e_j \in M} w(e_j)$$

be its *weighted matching polynomial* with its corresponding roots labelled as $\rho(m_w(G,x)) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The weighted matching polynomial is constructed from the groupings of k disjoint edges of G, called *k-matchings*. The largest such collection of disjoint edges is called a *maximum matching*.

In order to begin our study of the multiplicity of zero in $m_w(G,x)$, it is important to understand the properties of the roots of $m_w(G,x)$ and their multiplicities. It is a fun exercise to show that the roots of $m_w(G,x)$ are symmetric about zero [2], so the multiplicity of the root λ_i equals the multiplicity of the root $-\lambda_i$. To differentiate the precise ways in which multiplicities change as a result of vertex deletion, it can be shown that the multiplicity of any root of $m_w(G,x)$ can increase by one, decrease by one, or remain unchanged by removing a single vertex [2].

Definition 1. *Let G be a graph with weight function w and corresponding weighted matching polynomial, $m_w(G,x)$. For any vertex $v \in V(G)$, let $G \setminus \{v\}$ be a vertex-deleted subgraph of G with corresponding weighted matching polynomial, $m_w(G \setminus \{v\}, x)$. Suppose $\lambda_i \in \rho(m_w(G,x))$. A vertex v is said to be* upper with respect to $\pm\lambda_i$ (resp. downer with respect to $\pm\lambda_i$) *if the removal of v increases (resp. decreases) the multiplicity of $\pm\lambda_i$ by one. A vertex v of G is said to be* neutral with respect to $\pm\lambda_i$ *if the multiplicity of $\pm\lambda_i$ remains unchanged after the removal of v.*

Definition 2. *Let $G = (V, E, w)$ be a graph with a fixed weight function w and let $\lambda_i \in \rho(m_w(G,x))$. Then, every vertex in G lies in precisely one of the following classes corresponding to $\pm\lambda_i$:*

$$\text{Upper}(\pm\lambda_i) = \Big\{ v \in V \mid v \textit{ is upper} \Big\},$$

$$\text{Neutral}(\pm\lambda_i) = \Big\{ v \in V \mid \textit{ is neutral} \Big\}, \textit{ or}$$

$$\text{Downer}(\pm\lambda_i) = \Big\{ v \in V \mid \textit{ is downer} \Big\}.$$

For any graph G, the root, $\lambda_i = 0$, has a special relationship to the set of maximum matchings of G. If $\lambda = 0$ is a root of $m_w(G,x)$, then $m_w(G,x) = x^n - c_1 x^{n-2} + c_2 x^{n-4} - \cdots \pm c_k x^{n-2k}$, where k corresponds to the size of a maximum matching in G. Observe that each scalar c_i is necessarily nonzero for $i \in \{1, \ldots, k\}$ since the existence of a k-matching requires the existence of a $(k-s)$-matching for $s \geq 1$. Therefore, determining how the multiplicity of $\lambda_i = 0$ changes when a vertex v is removed relates precisely to the size of the

Jérémy Champagne
Co-Editor

Pour tirer le meilleur parti des connaissances acquises, pour en extraire toute la richesse, il importe de ne pas s'y habituer trop vite, de se laisser le temps de la surprise et de l'étonnement

– Hubert Reeves

Preamble

By: Jérémy Champagne (University of Waterloo)

Welcome to the Summer 2026 issue of Notes from the Margin, our 20th issue in total according to my (possibly faulty) bookkeeping. More importantly, this is my last one as Editor, as I have now lost my warm and comforting title of *student*, to instead join the uncertain world of postdoctoral ventures, sprinkled with a healthy dose of unemployment. For that same reason, I must also leave the CMS Student Committee. I don't want to reflect on it for too long, but what is clear is that I would not be the same mathematician if I had not applied to join the committee. It gave me such a broader view of the greater Canadian mathematical community, and I had the chance to meet the *people that make things happen*: journal editors, conference organizers, student leaders, committee chairs, plenary speakers, etc. I am proud of everything I have done through the committee, and I leave it as a significantly improved version of myself.

The sharpest of you might have noticed that I have been signing my emails as *Co-Editor* this semester. Indeed, I had the chance to work with Fateme Peimany who will take my place as Editor going forward, the same way I succeeded Courtney Allen when she graduated back in 2024. However, unlike myself, Fateme comes in with solid experience in publishing, so I think all of us should be very excited to see where she takes the journal! Please send her an article this Fall, there is no better way to welcome a new editor!

I am by no means the only committee member who is leaving this Summer. In this issue, you will find a short interview with two StudC *veterans* (Ludovick Bouthat and William Verreault) which I encourage you to read. On the mathematical side, you will also find articles involving graph theory, Euler's formula and some algebraic number theory. As usual, we have also received several short articles from the Women in Math DRP program at the University of Waterloo. Bonne lecture! ◄

[continued from cover]

maximum matchings in $G \setminus \{v\}$.

Theorem 1. *Let $\mathbb{M} = \{M_1, \ldots, M_s\}$ be the collection of maximum matchings for G of size k, and assume $0 \in \rho(m_w(G,x))$. Let $V(M_i)$ be the vertices induced by the edges of the matching M_i. Then,*

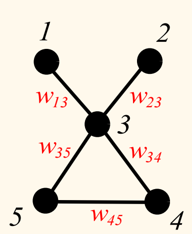
i) Upper$(0) = \bigcap_{i=1}^{s} V(M_i)$, and

ii) Downer$(0) = V \setminus$ Upper(0), and

iii) Neutral$(0) = \emptyset$.

Proof. Let G be a graph with n vertices. Suppose $\mathbb{M} = \{M_1, \ldots, M_s\}$ is a collection of maximum matchings for G of size k, and assume 0 is a root of $m_w(G,x)$ with $\text{mult}(m_w(G,x);0) = n-2k$. Thus, $m_w(G,x) = x^n - c_1x^{n-2} + c_2x^{n-4} - \cdots \pm c_kx^{n-2k}$. If the vertex v_i resides in every maximum matching in $\mathbb{M}$, consider $m_w(G \setminus \{v_i\}, x)$. Since every maximum matching contains an edge incident to v_i, the size of any maximum matching cannot exceed $k-1$ in $m_w(G \setminus \{v_i\}, x)$ and $m_w(G \setminus \{v_i\}, x) = x^{n-1} - d_1x^{n-3} + d_2x^{n-5} - \cdots \pm d_{k-1}x^{n-2k+1}$. Hence, v_i is upper with respect to 0, and $\bigcap_{i=1}^{s} V(M_i) \subseteq \text{Upper}(0)$. If the vertex, v_i, does not reside in every maximum matching in $\mathbb{M}$, then there exists a matching M_i of size k in both $m_w(G,x)$ and $m_w(G \setminus \{v_i\}, x)$, so $m_w(G \setminus \{v_i\}, x) = x^{n-1} - d_1x^{n-3} + d_2x^{n-5} - \cdots \pm d_kx^{n-2k-1}$. Hence, v_i is downer with respect to 0, and $\text{Downer}(0) \subseteq V \setminus \text{Upper}(0)$. Furthermore, the sets Upper(0) and Downer(0) contain all of the vertices in G, so $\text{Neutral}(0) = \emptyset$. □

Example 1. *Consider the following graph with fixed weight function, w. There are two maximum matchings: $M_1 = \Big\{\{1,3\},\{4,5\}\Big\}$ and $M_2 = \Big\{\{2,3\},\{4,5\}\Big\}$. It can be shown that Upper$(0) = \{3,4,5\}$ and Downer$(0) = \{1,2\}$.* ◄



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References

[1] C. Godsil, *Algebraic Combinatorics*. Chapman & Hall/CRC Mathematics Series, 1993.

[2] J. Parenteau, *Musings on Matchings, Matrices, and Multiplicities*. University of Regina Master of Science Thesis, 2024, University of Regina.

Interviews with some StudC veterans

By: Jérémy Champagne and Christine Eagles (University of Waterloo)

Since its creation in 1999, the Student Committee has had about a hundred[1] different members, at different points in times. It makes sense, since students have busy lives and can rarely serve more than a couple years on the committee. But, there are few members who serve the committee for longer, and they tend to have a large impact on the student body of the CMS. There are two such *veterans* who are leaving us this Summer, and we interviewed them for you! They both have been Chair of the committee in the past, and have organized plenty of events within and outside the CMS. Par souci de bilinguisme (on se rencontre au Nouveau-Brunswick cet été, après tout!), la deuxième entrevue est offerte en français.

William Verreault

University of Toronto – Member since 2020

How did you learn about StudC and the CMS in general? What made you want to join?

I was actually an undergrad when I first joined the CMS Committee. A prof I was working with, Javad Mashreghi, was already involved with the society (and soon to become president) at the time. He told me about the society and said that, if I intend to stay in academia, I might as well join now. He also told me that there is more to math than doing research or teaching, and one such aspect is giving back to the community. One way to do this as a student was to get engaged with the student committee. Part of the application form asked if we were interested in being a chair or co-chair and I thought: why not? Then the person they had selected for co-chair dropped out, so I ended up joining the committee as co-chair.

You joined the committee in 2020, at the beginning of the pandemic. Can you share the challenges that the committee faced during that period?

There were major budget cuts during COVID. One reason was because we were not using the money we would have used to fund student activities at CMS meetings. Certain things, like our student publication[a], were entirely dropped. It felt weird because, of course, the committee still cared and wanted to make things work, but at the same time we just didn't know what would happen. What do we do? How do we even communicate to new students who might be interested but don't know about the committee? The first question ended up being "*how do we survive?*". In the end, the CMS had four online meetings and surprisingly (or not), our best attendance for the student workshops and similar events were during those online meetings.

[a] which you are presumably holding right now.

What did you enjoy the most during your time on the committee?

My favourite part is the community and other members of the committee. Over six years, I have seen many members come and go, but some I have been to their weddings, I have visited in other cities. I even made my PhD application decisions based on other committee members. I don't think i could have done this on any other committee. My experience goes beyond what the committee was first meant to be.

What advice would you give to someone who is thinking of joining StudC? What advice would you have liked to receive before you joined?

The CMS runs on people. We need and want people with ideas, but also people who are willing to put in the work. Something I wish I knew before starting, is that this is different from some university committees where you meet maybe once or twice a semester and that's it. StudC could be like this, but the committee can be so much better when we all make an effort. In the end the members work for themselves, in the sense that they give to a community that they are a part of. Again, on those other committees you don't necessarily have much of an impact, but on StudC you do. The committee is you, your voice is important, what you do is important, because without it, it's nothing.

What accomplishment are you the most proud of?

During the pandemic, I proposed the Math Salon, a monthly zoom seminar where a student presented alongside their advisor. One would give the big picture of their field, and the other would dive into the student's research. There were even people from other disciplines that wrote to StudC saying that they also wanted to do this in their department. This ran for just over a year, but it felt like people benefitted from this. Partially separate from StudC, I was also on the committee from ULaval that re-started CUMC (in-person) after the pandemic. This is one of the things that took the most amount of time in my life to realise, but it was so rewarding. It gave a good example for the following years of CUMC. For myself having attended the 2019 CUMC, I knew what this experience should be, but would have been sad that future students would not know.

How does it feel to leave the committee after 6 years?

It is bittersweet. You know you need to leave at some point because it is a student committee, but I feel like it is a good time to leave. I have contributed a lot to the committee, but I feel like it can do well without me and it is time for my place to go to others. Throughout undergrad, master's and PhD, StudC has become part of my routine. But, I will still interact with StudC and stay involved with the CMS. I think it is part of this whole process that things evolve.

[1] In fact, we counted 101.

Ludovick Bouthat

Université Laval – Membre depuis 2021

Dans les trois dernières années tu as été président et co-président du comité. Quelle-est pour toi la plus grande différence au quotidien entre être président ou co-président?

La différence est énorme! Quand je suis tombé co-président (avec Kate), je savais que je ne serais pas aussi présent pour plusieurs raisons. J'ai pris le temps pour écrire un manuel à l'avance pour l'aider un peu, lui donner des ressources. Mais au final, Kate a été super bonne dans la transition. La présidence, c'est une grosse charge mentale, de s'assurer que tout le monde fasse ce qu'il doit faire. Être co-président, ça retire de la pression.

Quel conseil donnerais-tu à quelqu'un qui considère rejoindre StudC?

Ça serait juste d'y aller à fond. Surtout parce que c'est le fun (pour moi en tout cas, peut-être pas pour tout le monde). Ça peut faire peur: c'est le comité étudiant pour tout le Canada, c'est big! Tu perds rien à appliquer (c'est un petit formulaire) puis, si jamais tu es sur le comité et tu vois que ça ne marche pas, il n'y a rien de mal à faire juste un an et partir.

Qu'est-ce que tu as le plus apprécié de ton temps sur le comité?

Pour moi c'est clair, c'est les autres membres du comité, de rencontrer des gens. Plus généralement ce serait de rencontrer des gens au travers de la SMC, mais je ne serai jamais aussi proche que je l'ai été avec les autres membre du comité étudiant.

De quel accomplissement es-tu le plus fier?

Il y a eu plusieurs petits trucs du temps ou j'étais président. Plus personnellement, il y a la liste de conférences que j'ai ajouté à notre site web (ça faisait longtemps qu'on en parlait). On a aussi créer un manuel pour aider les nouveaux membres du comité. Sinon, il y a évidemment le retour de Notes from the Margin, mais ça c'est vraiment une victoire collective!

Comment décrirais-tu l'impact de StudC sur les étudiants au Canada?

Honnêtement: trop faible encore. On a déjà un bon impact avec Notes from the Margin et le financement qu'on donne aux conférences étudiantes (environs 5-6 par année), mais ça reste une lutte constante d'avoir de la visibilité dans tout le Canada. On voudrait avoir un plus gros impact, mais au final on reste des étudiants, on peut malheureusement pas s'impliquer à temps plein. Pour le futur, j'aimerais que le comité continu de s'aggrandir (pas nécessairement sur le nombre de membres), qu'on soit mieux connu, peut-être que le serveur Discord continu de grossir, des choses comme ça. Former une communauté.

Crois-tu que tu resteras impliqué dans la SMC après ta graduation?

Probablement que je continuerai à organiser des sessions de recherche aux rencontres de la SMC. En dehors de ça, il y a plusieurs autres comités avec des opportunités, il y a aussi le conseil d'administration mais c'est plus limité. J'aimerais bien être sur le conseil un jour!

Teaching Euler's Formula, Part 1: Understanding $e^{i\theta}$ as iterated multiplication

By: Jonathan Tot (Dalhousie University)

Last summer, as part of teaching an introductory ODEs course at Dalhousie University, I needed to review complex numbers for my students, including Euler's Formula for imaginary exponentials,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Students are frequently stymied in their endeavours to comprehend this strange expression—$e^{i\theta}$—what could possibly be understood by this purported computation? On this occasion, I had the strong desire to present this concept as *really* exponentiation, to whatever extent that may be possible. In this article I would like to share what insight I found.

Before diving right in, it will be helpful to review two other common proofs or derivations of 'Euler'. Probably the most common, and perhaps the most water-tight solution, is by power series. We take the exponential power series $e^z = 1 + z + z^2/2 + z^3/6 + \cdots = \sum_{k\geq 0} \frac{z^k}{k!}$, and evaluate with $z = i\theta$. This is often accompanied by *defining* the complex exponential function by this power series. The even terms are real with alternating sign, and together become precisely the power series for $\cos\theta$, while the odd terms are imaginary, also alternating signs, and we find precisely the series for $i\sin\theta$. Despite the relative ease and immediacy of this approach[1], it often leaves us feeling... well, wanting something more.

Jonathan Tot

Nothing takes place in the world whose meaning is not that of some maximum or minimum.

– Leonhard Euler

[1] If one cares to get into the gory details, all these series are absolutely converging, and so also unconditionally converging—this means the terms can be re-arranged in any order all down the sequence, with no change in the value of the final results.

Arguably, this is one of those cases in which a mechanical application of mathematics gets us the right answer, but it seems opaque and perhaps strange—what does any of this have to do with exponentiation?

Prior to what I have found recently, I would have said that my preferred derivation of the remarkable $e^{i\theta} = \cos\theta + i\sin\theta$ is via the differential property of the exponential function, $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$. Let me remark that in mathematics, when one wishes to extend a previously established concept to an extended domain of application, we often do so by highlighting certain properties of the established concept, and insist that they continue to hold in the extension. In order for this approach to succeed, these chosen properties need to uniquely determine the extended concept. These principles then come to form a property-based definition of the extended concept. As an example, this is how we extend the basic trigonometric ratios, from angles in triangles, to functions of any real value—by placing the concept of angle within the unit circle, and retaining the Pythagorean identity as a fundamental property.

Indeed, the function

$$E : \mathbb{R} \to \mathbb{C}, \quad E(\theta) = \cos\theta + i\sin\theta$$

already satisfies many of the properties of exponentiation that we would want to carry forward. Fundamentally, this is owing to the sum-of-angle trigonometric identities

$$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi,$$
$$\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi,$$

since from these we have not only

$$E(\theta)E(\varphi) = (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi) = E(\theta+\varphi) \quad (1)$$

but also, by repeated addition of the same angle,

$$E(\theta)^n = (\cos\theta + i\sin\theta)^n = E(n\theta), \; n \in \mathbb{Z}. \quad (2)$$

These are to be compared to the basic properties of exponentiation,

$$b^x b^y = b^{x+y}, \quad (3a)$$
$$(b^x)^y = b^{xy}, \quad \text{for any } b, x, y \in \mathbb{R}, \; b > 0, \quad (3b)$$

even though, to this stage, we can only parallel the (3b) property of real exponentiation to $y = n \in \mathbb{Z}$. Lastly, E also satisfies $E(0) = 1$, just as $b^0 = 1$ for all $b > 0$.

While all of this is great, it in no way confirms that our special function $E(\theta)$ is specifically the complex exponential $e^{i\theta}$. But, what if we regarded the result of differentiation[2],

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{kx} = k\,e^{kx} \text{ for any } k \in \mathbb{R}, \quad (4)$$

as a fundamental property of exponential functions? If we insist that our desired computation $e^{i\theta}$ observe this differentiation property, then $z(\theta) = e^{i\theta}$ must be the solution to the initial value problem

$$z' = \frac{\mathrm{d}z}{\mathrm{d}\theta} = i\,z, \quad z(0) = 1. \quad (5)$$

Now, the crux of this solution: break this out of the complex plane, turning the differential equation into a system for both parts of $z = x(\theta) + i\,y(\theta)$,

$$\begin{aligned} x' &= -y, \\ y' &= x \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} x'' &= -x, \\ y'' &= -y. \end{aligned}$$

Together with the initial conditions, it is an elementary exercise to confirm that the solution is $z(\theta) = \cos\theta + i\sin\theta$.

However, it is hard to regard that this argument demonstrates $e^{i\theta}$ as an exponential *arithmetically*; which is to say, as the result of a process of repeated multiplication. That is what I would like to present. Additionally, in order to be pedagogical, the derivations we have so far considered require students' prior knowledge of either Taylor series or differential equations, which typically come after at least one undergraduate calculus course. It would be of great value to be able to at least explain Euler's Formula to more junior students, perhaps familiar with only pre-calculus. It is for this reason that I will now turn to the main point of this article; the insight which came for the first time, at least to me, in Summer 2025. For this, we need to introduce the main characters of the show, which are the n^{th} roots of unity.

The n^{th} roots of unity are the solutions to the polynomial equation

$$z^n = 1 \quad (6)$$

for $n = 2, 3, 4, \cdots$. Unlike the case of solving more general polynomials over the complex numbers, it is elementary to solve for the roots of unity, and indeed we may employ the multiplication property (2) of $E(\theta)$ to do just that. However, first we need to establish the connection between our special function $E(\theta)$ and the n^{th} roots. We have that $E(-\theta) = \overline{E(\theta)}$ is the complex conjugate, so $|E(\theta)|^2 = E(\theta)\overline{E(\theta)} = E(\theta)E(-\theta) = E(0) = 1$, and thus we realize that the complex numbers $E(\theta)$ all have modulus or norm equal to 1; of course, the complex numbers $z = E(\theta)$ parameterize the unit circle in the complex plane, $|z| = 1$. Taking any root of unity, by the defining property (6) and a similar analysis, we have $|z|^n = |z^n| = 1$. Now, this equation has only one real, non-negative solution, so we find $|z| = 1$. By this we conclude that the roots of unity must be $z = E(\theta)$ for certain angles θ. From here, we have

$$z^n = E(\theta)^n = E(n\theta) = 1. \quad (7)$$

We easily identify that $E(\theta)$ is a periodic function, with period 2π, and that the only value $\theta \in [-\pi, \pi]$ such that $E(\theta) = 1$ is $\theta = 0$. The full solution of (7) is

$$E(n\theta) = 1 \quad \Longleftrightarrow \quad n\theta = 2\pi m \text{ for some } m \in \mathbb{Z}.$$

[2]The connection between the family of exponential functions, e^{kx} in (4), and b^x in (3), is by the logarithm, $k = \ln(b)$.

Keeping track of the periodicity of E, we see that the distinct n^{th} roots of unity are

$$z = E\big(2\pi\tfrac{m}{n}\big) \quad \text{for } m = 1, 2, \cdots, n.$$

We call the first solution $z = E\big(\frac{2\pi}{n}\big) = \cos\big(\frac{2\pi}{n}\big) + i\sin\big(\frac{2\pi}{n}\big)$ the *principal n^{th} root of unity*, and denote it by ω_n. The other n^{th} roots of unity are simply the integer powers of the principal root,

$$E\big(2\pi\tfrac{m}{n}\big) = \omega_n^m, \quad m = 1, 2, \cdots, n.$$

Having understood the complex roots of unity, I make the following claim: if we assume that $e^{i\theta}$ is 2π-periodic, and specifically that $\theta = 2\pi m$, $m \in \mathbb{Z}$, are the only solutions of $e^{i\theta} = 1$, then we must associate the roots of unity with certain of the purported quantities $e^{i\theta}$. We might take the premise as suggested from Euler's Formula itself, or we might be able to support it from some other intuition. As such, this is certainly not an independent proof or derivation of the Formula, but the main point is that it provides a further understanding of the imaginary exponentials $e^{i\theta}$, which I think is at least not commonly communicated.

Since $e^{i0} = e^0 = 1$, if $e^{i\theta}$ is 2π-periodic, then we identify the complex roots of unity as we maintain the (3b) property, since

$$\left(e^{i\,2\pi\frac{m}{n}}\right)^n = e^{i\,2\pi m} = 1 \quad \text{for } m \in \mathbb{Z}.$$

That $e^{i\,2\pi m} = 1$ are the only solutions assures us that $e^{i\,2\pi\frac{m}{n}}$ exhausts all of the n^{th} roots. Thus, these exponentials $e^{i\,2\pi\frac{m}{n}}$ are the roots of unity[3], and again, since $e^{i\theta}$ is 2π-periodic, we also deduce that the distinct solutions are those with $m = 1, 2, \cdots, n$.

From here, it is imperative that we go back and review how we actually understand (which is to say, how we compute) exponentiation for real exponents. Let me begin by assuming that we understand the computation of a^n for real-valued $a \geq 0$ and whole number $n \in \mathbb{N}$—this is of course simply n-times iterated multiplication with a,

$$a^n = 1 \cdot \underbrace{a \cdot a \cdots a}_{n}.$$

The first extension is to $n \in \mathbb{Z}$, specifically negative-integer exponents, using the reciprocal and the empty product (since $a^0 = 1$, then $a^{-1} = 1/a$ because $(1/a)a = 1$, and so on). Exponentiation by rationals is defined in terms of the n^{th} roots, $a^{1/n} = \sqrt[n]{a}$. This is fundamentally the solution to an inverse problem: $\sqrt[n]{a}$ is the non-negative real solution of $z^n = a$. Those who are interested may review the various algorithms available which compute the digits of the n^{th} root, given the digits of $a \geq 0$. Then, for rational $q = m/n \in \mathbb{Q}$, a^q is

$$a^q = (a^{1/n})^m = (a^m)^{1/n}.$$

Beyond the rationals, a real exponent x can be approximated by some sequence of rationals $\{q_k = m_k/n_k,\ m_k \in \mathbb{Z},\ n_k \in \mathbb{N}\}_{k=1}^{\infty}$ (e.g., the sequence of truncations of the digits of x, or perhaps the sequence of convergents), such that $\lim_{k\to\infty} n_k = \infty$ and $\lim_{k\to\infty} q_k = x$. Then, we define

$$a^x = \lim_{k\to\infty} a^{q_k}.$$

The reason to review all this is to realize that we can take the exact same step from the complex roots of unity to the exponentials $e^{i\theta}$ for real-valued θ. Specifically, suppose that $\theta/(2\pi)$ is approximated by some rational, m/n. Then we would approximate $e^{i\theta}$ by $e^{i\,2\pi\frac{m}{n}} = E(2\pi\frac{m}{n})$. We can confirm that $e^{i\theta}$ is close to an n^{th} root of unity:

$$\left(e^{i\theta}\right)^n \approx \left(e^{i\,2\pi\frac{m}{n}}\right)^n = e^{i\,2\pi m} = 1.$$

Completing the generalization, suppose that $\theta/(2\pi)$ is the limit of a sequence of rationals $\{q_k = m_k/n_k\}_{k=0}^{\infty}$,

$$\theta = 2\pi \lim_{k\to\infty} q_k.$$

Then we may compute $e^{i\theta}$ as

$$\begin{aligned} e^{i\theta} \longrightarrow \lim_{k\to\infty} e^{i\,2\pi q_k} &= \lim_{k\to\infty} \omega_{n_k}^{m_k} \\ &= \lim_{k\to\infty} E\left(2\pi\frac{m_k}{n_k}\right) \\ &= \lim_{k\to\infty} \left\{\cos\left(2\pi\tfrac{m_k}{n_k}\right) + i\,\sin\left(2\pi\tfrac{m_k}{n_k}\right)\right\} \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

The complex exponentials $e^{i\theta}$ can thus be understood as limits of sequences of complex roots of unity, which are given by the formula $\cos(2\pi q) + i\sin(2\pi q)$ for some rational q. These are distributed within the complex unit circle S^1 in just such a way as the rationals are embedded in the unit interval $[0, 1]$. I hope that this insight can be helpful to many students, whether reviewing Euler's Formula, or perhaps even learning complex numbers for the first time. In the next edition of the Notes, we will continue exploring along these lines, to arrive at a properties-based definition of $e^{i\theta}$ which requires neither power series nor differential calculus. ◀

[3] Eagle-eyed readers will notice the issue of matching between the roots of unity $\{E(2\pi\frac{m}{n}),\ m = 1, \cdots, n\}$ and the exponentials $\{e^{i\,2\pi k/n},\ k = 1, \cdots, n\}$, and indeed this is a point to be concerned with. Obviously, we want to make the identification $k = m$. We don't have the scope here to fully resolve this point, but a few things can be said. Given the 2π-periodicity assumption on $e^{i\theta}$, it will be seen that both sets manifestly have the cyclic group structure of $\mathbb{Z}/n\mathbb{Z}$. Then, any permutation $m = \sigma(k)$ between the two lists must be a group homomorphism. A consequence is that any viable matching must map those exponentials which can generate all others, with $\gcd(k, n) = 1$, to all those roots with $\gcd(m, n) = 1$. This can be characterized by the matching of $e^{i\,2\pi/n} \mapsto E(2\pi m^*/n)$ where $\gcd(m^*, n) = 1$. The only uniform options which exists, consistent m^* across all $n \geq 2$, are $m^* \equiv \pm 1 \bmod n$.

On the uniqueness of high school angles

By: Daniel Teixeira (Dalhousie University)

In high school I enjoyed a mnemonic for recalling the sines of 30°, 45°, and 60°: increasing as the angle increases, write down the numbers 1, 2, and 3, put square roots on each one, and divide them by 2. Since $\sin^2\theta + \cos^2\theta = 1$, the list of cosines is the same with the order reversed, and the tangents are recovered by dividing one by the other. Easy peasy, lemon squeezy!

Daniel Teixeira

"Wait a second, I know the synthesizer, why don't I use the synthesizer which *is* the sound of the future?"
Giorgio Moroder

The pattern also extends to 0° and 90°, i.e. ranging from 0 to 4 instead. The result:

angle	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sine	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

Neat! The expression for the angles in radians is sweet: rational fractions of π. Can we reproduce this mnemonic by trying out the same pattern for different numbers? That is, lay down numbers $\sqrt{k}/\ell$ for $0 \leq k \leq \ell^2$; are they sines of rational fractions involving π? For $\ell = 3$, we write our sines:

sine	$\frac{\sqrt{0}}{3}$	$\frac{\sqrt{1}}{3}$	$\frac{\sqrt{2}}{3}$	$\dots$	$\frac{\sqrt{8}}{3}$	$\frac{\sqrt{9}}{3}$

Looking at arcsin for these numbers shows no evident connection between the corresponding angles and π. (You can even try cubic roots.) So we try our luck and instead evaluate the sines of some rational fractions of π such as $\frac{\pi}{5}$, $\frac{7\pi}{9}$, $\frac{\pi}{17}$, but there is no clear pattern there either.

Question. Are there any other rational multiples of π whose sines are of the form $\sqrt{k}/\ell$ for natural numbers k and ℓ?

This question was also inspired by a post on Reddit [3], and here we answer it in the negative. A particular case is that the only first-quadrant rational multiples of π with rational sines are 0, $\pi/6$, and $\pi/2$, a fact known as *Niven's Theorem* [2].

We can approach the question via cosines. Recall the equation

$$\cos\theta = \frac{\exp(i\theta) + \exp(-i\theta)}{2}.$$

If the angle θ is of the form $2\pi n/m$ with $\gcd(n, m) = 1$, and $0 \leq n < m$, note that the numbers

$$\xi := \exp\left(\frac{2i\pi n}{m}\right) \quad \text{and} \quad \xi^{-1} = \exp\left(\frac{-2i\pi n}{m}\right)$$

are primitive m-th roots of unity.
We study the tower of field extensions

$$\mathbb{Q} \subseteq \underset{\mathbb{Q}(\cos\theta)}{\underset{\|}{\mathbb{Q}(\xi+\xi^{-1})}} \subseteq \mathbb{Q}(\xi).$$

Recall that the **degree** of a field extension $K \subseteq L$ is the dimension of L as a K-vector space and is denoted by $[L : K]$. Given a sequence of finite-dimensional extensions $F \subseteq L \subseteq K$, we have $[K : L] \cdot [L : F] = [K : F]$. [1, Chapter 14].

Lemma 1. $[\mathbb{Q}(\xi) : \mathbb{Q}(\xi + \xi^{-1})] \leq 2$ (with equality unless $\xi = \pm 1$).

Proof. Since ξ is a root of the quadratic polynomial $x^2 - (\xi + \xi^{-1}) \cdot x + 1$ with coefficients in $\mathbb{Q}(\xi + \xi^{-1})$, the degree is 2 or less. □

Lemma 2. $[\mathbb{Q}(\xi) : \mathbb{Q}] \geq \varphi(m)$ (where φ is the Euler totient function)

Proof. Since $\mathbb{Q}(\xi)$ is the splitting field of the separable polynomial $x^m - 1$, the extension is Galois, and the degree is the order of its Galois group. If a is a positive integer with $\gcd(a, m) = 1$, then $\xi \mapsto \xi^a$ defines an automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$, so the Galois group has at least $\varphi(m)$ elements (one for each a). □

(In fact $[\mathbb{Q}(\xi) : \mathbb{Q}] = \varphi(m)$, but we only need the bound.)

Corollary. $[\mathbb{Q}(\cos\theta) : \mathbb{Q}] \geq \varphi(m)/2$.(with equality unless $\theta = 0, \pi$)

If moreover $\cos\theta$ is of the form $\sqrt{k}/\ell$, then $[\mathbb{Q}(\cos\theta) : \mathbb{Q}] \leq 2$, so with the corollary this assumption gives $\boxed{\varphi(m) \leq 4}$.

As $\varphi(m)$ counts the coprime positive integers less than m, we can inspect by hand the solutions to $\varphi(m) \leq 4$, which are exactly 1, 2, 3, 4, 5, 6, 8, 10, and 12. More directly, when factoring m as $m = p_1^{a_1} \cdots p_r^{a_r}$, we have the identity

$$\varphi(m) = p_1^{a_1-1}(p_1 - 1) \cdots p_r^{a_r-1}(p_r - 1),$$

and solving $p^{a-1}(p - 1) = 1, 2, 4$ yields the possible choices of p and a.

We can restrict ourselves to first-quadrant angles. The cases $m = 1, 4, 6, 8, 12$ contain only our familiar solutions:

$$\theta \in \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right\}.$$

The cases $m = 2, 3$ contribute to no first-quadrant angles, so we can ignore them. On the other hand, $m = 5, 10$ correspond to $\cos(2\pi/5), \cos(\pi/5) = (\sqrt{5} \mp 1)/4$, which is not of the form $\sqrt{k}/\ell$, and with this we exhaust our possibilities.

Remark. Galois Theory can't exclude the cases $m = 5, 10$: it only sees the degree of the extensions, not the specific form of the algebraic number being adjoined.

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Women in Math:

Directed Reading Program Articles

For me, the formula for math research has often been unclear. Unlike in other STEM fields, math research doesn't explode from a beaker, squeeze into a pipette, or reveal itself under a microscope. As an undergraduate, I couldn't even picture what generating new math looked like. In class, I learned about classical results from mathematicians who didn't look like me, and on homework, I proved statements I already knew were true. While I now have first-hand research experience as a grad student, other questions have emerged. Once I graduate, how will I choose a new question to work on, and how will I guide my (future) students through dissertations of their own? Luckily, the Directed Reading and Research Program (DRP) at the University of Waterloo is in the business of demystifying math research for undergraduates and graduate students alike.

Organized by Waterloo's Women in Mathematics (WiM) Committee since 2022, the DRP pairs graduate students and their proposed projects with undergraduates who are underrepresented in math for a semester. In the DRP-Reading stream, mentors guide students through a topic not usually covered in class. WiM also launched the DRP-Research stream in 2024 due to popular demand. These teams generate novel results, giving mentees a taste of what math research really looks like.

Scan to learn more about the WiM DRP and watch a DRP video.

While the DRP model originated at the University of Chicago and is now an active initiative at over twenty universities, Waterloo's version has a twist. First, by reserving mentee spots for students who identify as underrepresented in math, our DRP is likely the first to incorporate equity, diversity, and inclusion into the program's makeup. It's also the first faculty-wide DRP, supporting over 30 projects per semester across Waterloo's five math departments. Finally, the delineation between reading and research projects provides opportunities for undergrads at all stages, encouraging them to stick with math. Graduate students also get the unique experience of serving as mentors before they are in a faculty position.

As a DRP mentor, I proved to myself that becoming a supervisor one day may not be so far-fetched. Meanwhile, I saw my mentees grow confident not only in their understanding that math research is the act of constructing new results from the building blocks of what's known, but also in their ability to contribute to this tower of knowledge.

Read on to find accessible introductions to some of our recent DRP reading and research projects.

— Cicely (Cece) Henderson (University of Waterloo)

How cells feel: modeling cell interactions with their environment

Applied Math

By: Kris Zhang, Margaret A. Puzio, and Gordon R. McNicol (University of Waterloo)

Cells do not just respond to chemical signals – they can also "feel" their environment. By sensing whether their surroundings are soft or stiff, and by responding to forces exerted by these surroundings, cells regulate how they move, divide, and change shape. This process, known as *mechanotransduction*, enables essential bodily functions such as touch, hearing, and balance, and is implicated in diseases including cardiovascular disorders and cancer.

Our project developed a mathematical model describing how cells assemble two key mechanotransducing structures:

- **Stress fibers** are contractile bundles of cross-linked filaments that behave similarly to muscle fibers and form a major component of the cell cytoskeleton.
- **Focal adhesions** connect these internal fibers to the cell's external environment, stabilizing cell shape and transmitting forces that trigger biochemical signaling.

The development of these two structures is connected through a positive feedback loop: focal adhesion maturation promotes stress fibre assembly, which in turn reinforces focal adhesions.

To describe this process quantitatively, we used the law of mass action to develop a system of 18 ordinary differential equations. The model tracks key protein concentrations (e.g. actin and myosin, components of stress fibers) and their interactions over time. While some reaction rates were held constant, others depended on protein signals to capture feedback between biochemical processes and force generation.

Model simulations reproduced observed protein dynamics and revealed two distinct time scales: a fast initial phase when weak adhesions form upon first contact with their environment, and a slower phase where feedback drives the maturation of stress fibers and focal adhesions. A sensitivity analysis identified parameters which most strongly affect the speed and extent of structure formation, crucial because some biochemical rates are poorly measured experimentally. Finally, simulations showed how specific inhibitors (including drugs used clinically for hypertension) can disrupt the feedback loop, causing mechanosensing structures to break down.

By linking biochemical signaling and mechanical force within a reduced framework, the model balances biological realism with mathematical simplicity. It provides a systematic way to explore feedback-driven mechanosensing and to test predictions that are difficult to probe experimentally, offering insights into how cells behave in both healthy and diseased states. More broadly, this work advances our understanding of how cells integrate mechanical and biochemical cues, with implications for tissue engineering, disease modeling, and therapeutic design. ◂

Extending Positive Interior Quadrature Rules to Higher Degrees

Applied Math

By: Moustapha Diallo, Chloe Young, and Zelalem Arega Worku (University of Waterloo)

Quadrature rules, which approximate integrals by weighted sums, are indispensable in numerical methods for partial differential equations. In high-order discretizations such as finite element and discontinuous Galerkin methods, accuracy and stability depend on the exactness and structure of quadrature rules employed. Classical Gaussian quadrature achieves optimal accuracy in one dimension [1], but extending such efficiency and positivity to higher dimensions is nontrivial. To this end, we construct *positive interior* (PI) quadrature rules, where all nodes lie strictly inside the domain and all weights are positive. These rules are valuable for discrete conservation, preventing spurious oscillations, and enabling matrix-norm stability estimates. A general rule approximates an integral over a reference domain Ω as

$$\int_\Omega f(\mathbf{x})\,\mathrm{d}\Omega \approx \sum_{i=1}^{n_q} w_i f(\mathbf{y}_i),$$

with $\mathbf{y}_i$ and w_i denoting nodes and weights. The rule is exact for all polynomials $\mathcal{P} \in \mathbb{P}_q(\Omega)$ of degree $\leq q$ if the discrete moment conditions

$$\mathbf{g} := \mathsf{V}^\top \mathbf{w} - \mathbf{f} = \mathbf{0}$$

are satisfied, where V is the Vandermonde matrix of basis evaluations and $\mathbf{f}$ the vector of exact basis integrals. In practice, we solve the nonlinear least-squares problem

$$\min_{\boldsymbol{\tau}} \ \tfrac{1}{2}\,\mathbf{g}^T\mathbf{g},$$

with $\boldsymbol{\tau}$ collecting node coordinates and weights associated with unique symmetry orbits.

Constructing high-degree PI rules is notoriously challenging because both the nodes and weights are unknown and highly sensitive to initialization, with poor guesses leading to divergence or negative weights. To overcome this, we developed novel initial guess strategies and node elimination techniques that reduce node counts while preserving accuracy, symmetry, and positivity. Our new quadrature rules reach degrees up to 47 on the quadrilateral, 23 on the hexahedron, 21 on the prism, and 17 on the pyramid, all with positive weights, interior nodes, and efficiency matching or exceeding existing results. Ongoing work targets higher-degree rules and further node reduction, expanding feasible high-order discretizations on mixed-element meshes and strengthening the foundation for accurate and stable numerical methods. ◀

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Exploring Social Networks Through c-Closed Graphs

Combinatorics and Optimization

By: Gabriela Bourla, Shruthi Konduru, Kiera Mitchell, and Gul Rukh (University of Waterloo)

Social networks are structures that represent interactions between parties, and studying them can help discover more about a population and the dynamics within it, whether that consists of in-person friendships or online ones through social media. A common way to model social networks is through graphs where a person is represented by a dot, which we call a vertex, and friendship between two people is expressed as a line drawn between them, which we call an edge. Two vertices with an edge between them are called adjacent, so a pair of adjacent vertices represents two friends.

In our project, we studied Fox, Roughgarden, Seshadhri, Wei, and Wein's 2020 paper [1] in which they define a deterministic property for graphs representing social networks. Earlier work used generative models that described how vertices join the graph. Instead, this new paper focused on the property of *triadic closure*: the idea that two people with mutual friends are more likely to be friends themselves. We can make this concept more concrete by choosing a positive integer c and enforcing that if two people have at least c mutual friends, then they must also be friends with each other. This can be defined in graph theory terms through the idea of neighbors, where the neighbors of a vertex v are all the vertices that are adjacent to v. Fox, Roughgarden, Seshadhri, Wei, and Wein defined a *c-closed graph* to be a graph where any two vertices that have at least c neighbors in common must be adjacent.

Friend groups in social networks can also be represented through graphs. A *clique* is a part of a graph where every pair of vertices has an edge between them. We call a clique *maximal* if no other vertex in the graph is adjacent to every vertex in the clique, so no vertex can be added to keep it a clique. The paper considered the problem of counting the maximal cliques in a graph. Past results showed that in general, the upper bound on the number of maximal cliques in a graph is exponential in n, the number of vertices. Fox, Roughgarden, Seshadhri, Wei, and Wein found that, for c-closed graphs, this problem is *fixed-parameter tractible* in c. This means that the number of maximal cliques is polynomial in n and exponential in c, which will not get too big. For our project, we read through the proofs in the paper that found upper and lower bounds on the number of maximal cliques. The proofs often used the property that in a c-closed graph, non-adjacent vertices must have fewer than c common neighbors. ◀

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Digital Civics and Community Engagement with Technology

Computer Science

By: Angela Li, Erin Walshaw, Alexandra Roszczenko, Anthony Maocheia-Ricci (University of Waterloo)

Digital civics as a research field bridges the gap between political science and computer science. Much technology in this space supports *community* or *civic engagement* both between community members and researchers, or among community members themselves. Where much work exists in this field, we wish to define a set of simple design guidelines for community engagement technology – rooted in literature. As such, our DRP project involved exploring social computing research in digital civics and community engagement to then propose a set of 3 broad design guidelines (D1-D3) to create effective technological systems for community engagement:

D1: The system should encourage and enforce collaboration with community members whenever possible. Platforms to support community engagement should focus on collaboration rather than only contribution [4]. Thus, these platforms should support involvement through question formation, co-design, data analysis, prototyping, and evaluation with respect to the problem at hand.

D2: The system should support community members by enabling accessible, sustainable, and accountable civic engagement avenues. Platforms should offer various mediums for communication by community members, keeping in mind what digital tools are already in use and complement them [2]. Barriers to participation, such as accessibility concerns, should be managed by the platform itself [1].

D3: This system should empower community members by upholding their values and encouraging diverse perspectives and beliefs. Technological systems are not neutral by design [3]. As such, platforms created for community engagement should recognize this non-neutrality or *bias*, and prioritize civic values and trust. ◄

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The Ramsey theorem with ultrafilters

Pure Math

By: Jashan Bal, Yanna Jaskielewicz, and Megan Winterburn (University of Waterloo)

If S is an infinite set and $S = S_1 \sqcup S_2$ is a partition, then one of the S_i's must also be infinite. This somewhat obvious property is known as the infinite Pigeon hole principle and motivates the study of Ramsey type properties. Let $[\mathbb{N}]^2 = \{\{x, y\} : x, y \in \mathbb{N} \text{ distinct}\}$ be the edges of the complete graph on vertex set $\mathbb{N}$. The infinite Ramsey theorem states that if $\gamma : [\mathbb{N}]^2 \to \{1, 2\}$ is a red/blue colouring, then there exists an infinite subset $M \subseteq \mathbb{N}$ such that the induced complete subgraph on M is monochormatic (all the edges have the same colour). We now present ultrafilters, a tool for proving the infinite Ramsey theorem.

A *filter* $\mathcal{F}$ on $\mathbb{N}$ is a nonempty collection $\mathcal{F}$ of subsets of $\mathbb{N}$ such that $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$, and $\mathcal{F}$ is upwards closed. Here upwards closed just means that if $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$. For example, consider the Fréchet filter which is the set of all cofinite subsets of $\mathbb{N}$, i.e., $\{S \subseteq \mathbb{N} : \mathbb{N} \setminus S \text{ finite}\}$. An *ultrafilter* is a filter $\mathcal{U}$ with the additional property that for any $A \subseteq \mathbb{N}$ we have that either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$. Given any $n \in \mathbb{N}$ we have that $\{S \subseteq \mathbb{N} : n \in S\}$ is an ultrafilter, known as a principal ultrafilter. Using the Axiom of Choice one can show the existence of non-principal ultrafilters: by Zorn's lemma take a maximal filter extending the Fréchet filter and then show it has to be a non-principal ultrafilter. The key observation is that every set in such a filter has to be infinite.

Here is a brief sketch for the proof of the infinite Ramsey theorem. Fix some red/blue colouring of $[\mathbb{N}]^2$. For each $n \in \mathbb{N}$ we can partition $\mathbb{N}$ into the vertices R_n which share a red edge with n, the vertices B_n which share a blue edge with n, and n itself. Now we must have that either $R_n \in \mathcal{U}$ or $B_n \in \mathcal{U}$. We shall say that n is $\mathcal{U}$-red if $R_n \in \mathcal{U}$ or is $\mathcal{U}$-blue if $B_n \in \mathcal{U}$. It follows that we can also partition $\mathbb{N}$ into the $\mathcal{U}$-red and $\mathcal{U}$-blue vertices. Once again as $\mathcal{U}$ is an ultrafilter one of these sets must belong to $\mathcal{U}$. Without loss of generality suppose $A := \{n : n \text{ is } \mathcal{U}\text{-red}\} \in \mathcal{U}$. Now using that $\mathcal{U}$ is closed under intersections inductively pick $n_1 \in A$, $n_2 \in A \cap R_{n_1}$, $n_3 \in A \cap R_{n_1} \cap R_{n_2}$, and so on. Non-principality of $\mathcal{U}$ insures we can pick distinct n_i's. We obtain that $\{n_1, n_2, \dots\}$ forms an infinite complete monochromatic subgraph.

There are many more Ramsey type results we can prove with the use of what are know as idempotent ultrafilters and minimal idempotent ultrafilters! See [2] for more on this. ◄

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Introduction to Semigroup Theory for Differential Equations

Pure Math

By: Shri Lal Raghudev Ram Singh, Elisa Liang & Alice Shimiao Zhang (University of Waterloo)

At the heart of the theory of differential equations (DEs) lies answering three fundamental questions: *Does a solution exist? Is it unique? And does it depend continuously on the initial data?*. If the answer to the above questions is yes, then a given ordinary differential equation (ODE) or a partial differential equation (PDE) is said to be *well-posed*.

It is easy to see that the solution map $\mathbf{u} : [0,\infty) \to \mathbb{R}$ of the scalar ODE of the form $\dot{\mathbf{u}}(t) = a\mathbf{u}$, $a \in \mathbb{R}$ with initial value $\mathbf{u}(0) = u \in \mathbb{R}$ is given by $\mathbf{u}(t) = e^{at}u$ (*well-posed*!). Interestingly, the family $\{e^{at}\}_{t\geq 0}$ of bounded linear maps from $\mathbb{R}$ to $\mathbb{R}$, $\forall\, t \geq 0$, enjoys the algebraic properties of a semigroup under composition; that is, $e^0 = 1$ and $e^{a(t+s)} = e^{at} \cdot e^{as}$. Therefore, for *well-posedness*, one might expect similar properties to hold while studying DEs on infinite-dimensional abstract function spaces. This is precisely where the elegance of semigroup theory comes into play! In particular, we study the existence and uniqueness of a solution $\mathbf{u} : [0,\infty) \to X$, where X is a real Banach space, of the first-order ODE (or a PDE formulated as an ODE)

$$\begin{cases} \dot{\mathbf{u}}(t) = A\,\mathbf{u}(t), & (t \geq 0), \\ \mathbf{u}(0) = u, \end{cases} \tag{1}$$

where $u \in X$ is given and A is a linear operator with domain $D(A)$. In this project, we reviewed functional analysis and semigroup methods for DEs. Mathematically, a *C_0-semigroup* $\{S(t)\}_{t\geq 0}$ on X is a family of bounded linear operators satisfying: **i)** $S(0) = I$; **ii)** $S(t+s) = S(t)S(s)\ \forall\, t, s \geq 0$; **iii)** the mapping $t \mapsto S(t)u$ is continuous from $[0,\infty)$ to X. If, in addition, $\|S(t)\| \leq 1\ \forall\, t \geq 0$, then $\{S(t)\}_{t\geq 0}$ is a *semigroup of contractions*. We define $D(A) := \{u \in X \,|\, \lim_{t\to 0^+}(S(t)u - u)/t \text{ exists in } X\}$ and $Au := \lim_{t\to 0^+}(S(t)u - u)/t$ for $u \in D(A)$. Then $A : D(A) \to X$ is called the *infinitesimal generator* of the semigroup $\{S(t)\}_{t\geq 0}$. Therefore, if A in (1) is the *infinitesimal generator* of a *C_0-semigroup* $\{S(t)\}_{t\geq 0}$, then the solution can be written as $\mathbf{u}(t) = S(t)u$, and the ODE (1) is *well-posed*.

At this point, a natural question arises: what hypotheses on A are sufficient for it to generate a *C_0-semigroup of contraction*? This is answered by the *Hille-Yosida theorem* [1], which states that a linear (possibly unbounded) operator A generates a *C_0-semigroup of contractions* if and only if A is a densely defined closed operator and its resolvent set contains $(0,\infty)$ with the resolvent estimate $\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$, $\forall\, \lambda > 0$. ◄

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Considering Climate Impact in Optimizing Investment Portfolios

Statistics and Actuarial Science

By: Rhoda Dadzie-Dennis, Minh Chau Nguyen, Milagro Chen, Suneet Kaur, Shreya Jain (University of Waterloo)

Climate change is not only an environmental concern but also a financial risk due to its impacts on assets. As climate risks intensify, investors must balance traditional goals in investment with Environmental, Social, and Governance (ESG) considerations. However, fund managers may still be hesitant to include ESG in their investing strategy for fear of compromising their shareholders' benefits. While a majority of empirical evidence suggests a positive correlation between the ESG performance of a portfolio and their financial performance [1], such relation depends heavily on the asset universe and on the detailed portfolio optimization framework [2]. In this project, we aim at obtaining our own conclusion on how ESG considerations affect portfolio risk and return by carrying out an experiment.

As part of this process, ESG score is an important tool to quantify the climate and societal impacts of firms' operation and their exposure to climate-related risks; both with direct implications on their stocks' performance. As such, the ESG performance of a portfolio can be measured as a weighted sum of the stocks' ESG score. To provide a comprehensive result, we consider different climate scenarios ranging from relatively stable future to severe climate stress, respectively associated with tighter ESG requirements, which are then added to traditional mean-variance optimization and Sharpe ratio maximization frameworks. At a glance, imposing ESG requirements limits the set of admissible assets, which may alter both the expected return and the overall risk of the optimal portfolio in unexpected ways.

Findings include that moderate ESG constraints often have little effect on optimal portfolios. However, under more extreme climate scenarios, stricter constraints significantly reduce the feasible set of assets, leading to lower risk-adjusted returns and increased volatility.

In simple terms, sustainable investing does not necessarily require sacrificing performance, at least not initially. But as constraints become more stringent, the trade-off becomes unavoidable. Understanding where this balance lies is essential as financial markets continue to adapt to a changing climate. ◄

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About the StudC

The CMS Student Committee (StudC) was created in 1999 to help the Society better serve the needs of university students in mathematics. We organize activities *for students, by students*! This includes regular events at the biannual CMS Meetings, but also other initiatives such as our student publication (currently in your hands), supervising the CUMC bidding process and granting funding for regional conferences. Being part of StudC means developing skills in leadership, communication, teamwork, event planning, writing, outreach, and more. It's a great space to develop professionally and give back to your community, but it's also a lot of fun in itself!

If you have any question about StudC, feel free to contact our chair at **chair-studc@cms.ca**, or you can simply visit our website via the QR code at the (very) bottom of this page. We also encourage you to join our Discord server which we use to make regular announcements and communicate with you folks! ◀

Join us on Discord!

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