Image credit: Philippe-André Luneau

# Notes from the Margin

Volume XV · 2023

## The Loop Group

By: Justin Fus (University of Toronto)

As with most fields of mathematics, our discussion is motivated by the structure of $\mathbb{R}$ and $\mathbb{R}^n$. It seems to be such a naturally convenient structure that can be viewed as both a group and a topological space. More specifically, it can be regarded as a group and a smooth manifold.

Being mathematicians, $\mathbb{R}^n$ feels a bit too specific and lacks a sense of infinitude, and so we generalize. Of course, in order to put together a notion of smoothness, we need a "calculus on infinite-dimensional space".

**Definition.** *Let $E$ be a topological vector space and $U \subseteq E$ an open subset. A map $f : U \to E$ is $C^1$ if*

$$Df(u; v) = \lim_{t \to 0} \frac{f(u + tv) - f(u)}{t}$$

*exists for all $u \in U$ and $v \in E$ and the map $Df : U \times E \to E$ is continuous. The map $Df$ is meant to take a point in $U$ and a direction $v \in E$ in which we take our derivative. $f$ is $C^2$ if*

$$D^2 f(u; v_1, v_2) = \lim_{t \to 0} \frac{Df(u + tv_2; v_1) - Df(u; v_1)}{t}$$

*exists for all $u \in U$, and $v_1, v_2 \in E$, and $D^2 f : U \times E \times E \to E$ is continuous. Similarly, $D^2 f$ involves starting at $u$, differentiating in the direction of $v_1$, and then differentiating this directional derivative in the direction of $v_2$. We can then extend these classes of functions. For $n > 1$, a function is $C^n$ if $Df$ is $C^{n-1}$. $f$ is then* ***smooth*** *(or $C^\infty$) if it is $C^n$ for all $n \geq 1$.*

A smooth manifold modelled on a topological vector space $E$ is then defined analogously to its finite-dimensional case with $\mathbb{R}^n$ replaced by $E$. A **Lie group** $G$ is then a smooth manifold such that the group operation $G \times G \to G$ and inversion $G \to G$ are smooth maps.

The unit circle $S^1$ is an example of a Lie group. Viewing $S^1$ as complex numbers $e^{i\theta}$ of modulus one, elements of the circle are now identified with rotations. This gives the circle the structure of a group (of rotations) while also being a smooth manifold.

Infinite-dimensional Lie groups are not too far out of reach. If $X$ is a compact space and $G$ is any finite-dimensional Lie group, then the space $\mathrm{Map}_{\mathrm{cts}}(X; G)$ forms an infinite-dimensional Lie group. The topology we endow this space with is that of uniform convergence. This means, loosely, that a sequence of maps $f_n$ converges to $f$ if and only if outputs of $f_n$ at each point converge to outputs of $f$ "all at once". The group operation is pointwise, meaning we combine $f$ and $g$ so that $(fg)(x) = f(x) \cdot g(x)$ combines outputs of $f$ and $g$ to give the output of $fg$. An interesting choice of $X$ is the unit circle $S^1$ where we obtain the loop space of $G$.

We defined a notion of smoothness for smooth manifolds, so we can consider another related space that deals with smooth maps from the circle.

**Definition.** *Let $G$ be a Lie group. The* ***loop group*** *$LG$ of $G$ is the space of smooth maps $\gamma : S^1 \to G$.*

The loop group of a Lie group forms an infinite-dimensional Lie group.

**Example.** *Consider the Lie group $G = GL_2(\mathbb{R})$ of $2 \times 2$ real invertible matrices. Treating $S^1$ as the set of complex numbers with unit modulus, an example of an element of $LG$ is*

$$\gamma(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

[Continued on page 3]

Justin Fus

I have a bit of an obsession with anything and everything infinite. It's like an all-you-can-eat mathematical buffet for the mind. It is in these instances that you realize how finite your stomach truly is.

# Preamble

By: Courtney Allen (University of Guelph)

Courtney Allen
Editor-In-Chief

What does it take to be a mathematician? [...] It does not take brilliance, but love of a great game!
- Karen Keskulla Uhlenbeck

As I've been approaching the end of my PhD, I've started thinking about what comes after all of this schooling. Should I stay in academia? Or maybe I want to branch out into industry or government? How can I use my math degree in a way that is both impactful and personally fulfilling? Wherever you are in your education, I'm sure you've thought about these things too. Mathematics is such a vast field of study that thinking about making it a career can feel overwhelming.

I wanted to introduce students to possible career paths they might not have considered, so I reached out to a former editor of *Notes from the Margin*, Kseniya Garaschuk. She put me in touch with Dr. Megan Dewar, the head of the Tutte Institute for Mathematics and Computing. Dr. Dewar was kind enough to talk with me about her experiences working in the public sector with a math degree. Our conversation can be found on page 10 of this issue, and I hope it proves as interesting to you as it was to me.

If career-talk isn't for you though, don't fret! We also have interesting articles on group theory, Fourier series, and topology. Are you thinking of attending the Canadian Undergraduate Mathematics Conference? Check out the article detailing the CUMC experience. Looking to flex your mathematical muscles? Try your hand at the Winter Puzzles. There's something for everyone in this jam-packed edition of *Notes from the Margin*!

If you want to see your work in *The Margin*, or if you have questions or comments about the articles in this issue, contact the editor at **student-editor@cms.math.ca**.

[Continued from cover page]

There is a subgroup of the loop group called the **based loop group** consisting of maps $\gamma \in LG$ such that $\gamma(1) = e_G$ is the identity element of $G$. This space is usually denoted $\Omega G$.

To top off our little survey of some infinite-dimensional Lie groups, let $X$ be a compact smooth manifold. The **diffeomorphism group** $\mathrm{Diff}(X)$ of $X$ is the space of diffeomorphisms $f : X \to X$. This space interacts nicely with the loop group. In much the same way that the group of matrices can act on vectors by matrix-vector multiplication, diffeomorphisms $f$ of $S^1$ can act on a loop $\gamma \in LG$ by first smoothly deforming the circle with $f$ and then applying our loop $\gamma$.

**Proposition.** $\mathrm{Diff}(S^1)$ *has a natural right action on* $LG$.

The action is defined, like described, as

$$(\gamma \cdot f)(z) = \gamma(f(z))$$

We have

$$((\gamma \cdot f) \cdot g)(z) = (\gamma \cdot f)(g(z)) = \gamma(f(g(z))) = (\gamma \cdot fg)(z)$$

and

$$(\gamma \cdot \mathrm{id}_{S^1})(z) = \gamma(z).$$

## References


[1] A. Pressley, G. Segal *Loop Groups* Oxford Mathematical Monographs, 1986.

[2] A. Kriegl, P. W. Michor *The Convenient Setting of Global Analysis* Mathematical Surveys and Monographs, 1997.


# In This Issue:

# Phénomène intriguant de Gibbs

By: Pierre-Olivier Parisé (University of Hawaii)

**Les séries de Fourier sont beaucoup utilisées pour résoudre des équations différentielles. Elles sont aussi utilisées pour approximer les valeurs de certaines fonctions. Cependant, lorsque notre fonction ne se comporte pas gentiment, un drôle de phénomène se produit qui rend impossible l'approximation dans certains cas!!**

Ce phénomène s'appelle le phénomène de Gibbs. Pour bien le comprendre, nous présentons le concept de séries de Fourier et de ses sommes partielles associées. Bien sûr, plusieurs mathématiciens ont pensé à une façon de remédier à ce problème. Nous allons présenter une méthode qui utilise les moyennes de Cesàro.

## 1. Séries de Fourier

Joseph Fourier a été un des premiers à revendiquer la possibilité d'exprimer une fonction $2\pi$-périodique $f : \mathbb{R} \to \mathbb{R}$ par une série trigonométrique. Une série trigonométrique est de la forme

$$a_0 + \sum_{n\geq 1} a_n \cos(nt) + b_n \sin(nt). \qquad (1)$$

Cette serie est appelée de nos jours *série de Fourier* en son honneur. Mais, vous me direz, comment cette dernière expression est-elle reliée à la fonction $f$?

Assumons que la fonction $f$ coïncide avec (1), multiplions cette dernière équation par $\cos(kt)$ ($k \geq 1$ un entier) et intégrons le résultat de 0 à $2\pi$. On obtient donc

$$\int_0^{2\pi} f(t)\cos(kt)\,dt$$
$$= \int_0^{2\pi} \Big(a_0 + \sum_{n\geq 1} a_n \cos(nt) + b_n \sin(nt)\Big) \cos(kt)\,dt.$$

En supposant qu'il est possible d'interchanger les symboles d'intégration et de sommation, le membre de droite de la dernière relation équivaut à

$$\int_0^{2\pi} a_0 \cos(kt)\,dt + \sum_{n\geq 1} a_n \int_0^{2\pi} \cos(nt)\cos(kt)\,dt \qquad (2)$$
$$+ b_n \int_0^{2\pi} \sin(nt)\cos(kt)\,dt$$

Nous avons les identités suivantes:

$$\int_0^{2\pi} \cos(nt)\cos(kt)\,dt = \pi\delta_{n,k}$$

et

$$\int_0^{2\pi} \sin(nt)\cos(kt)\,dt = 0, \qquad (n \neq k).$$

où $\delta_{n,k}$ est le delta de Kronecker qui vaut 0 lorsque $k \neq n$ et 1 lorsque $k = n$. Ainsi, en remplaçant ces dernières identités dans (2), on obtient la relation suivante:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt. \qquad (3)$$

L'expression (3) est appelée le *coefficient en cosinus de Fourier* de la fonction $f$.

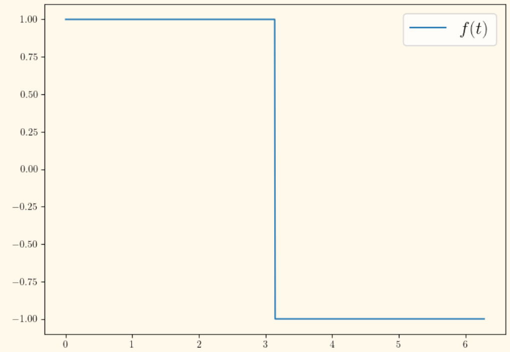



Figure 1: Graphe de la fonction $f$

On peut répéter l'exercice en multipliant (1) par $\sin(kt)$ ($k \geq 1$ un entier) et en intégrant le résultat de 0 à $2\pi$. En utilisant l'identité

$$\int_0^{2\pi} \sin(nt)\sin(kt)\,dt = \pi\delta_{n,k}$$

on obtient l'expression du coefficient $b_n$ en terme de la fonction $f$ :

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt \qquad (4)$$

Qu'en est-il du coefficient $a_0$? Pour obtenir son expression, il suffit d'intégrer (1) de 0 à $2\pi$ directement. Comme les intégrales faisant intervenir les fonctions $\cos(nt)$ et $\sin(nt)$ sont nulles, on obtient

$$\int_0^{2\pi} f(t)\,dt = a_0 \int_0^{2\pi} dt = 2\pi a_0.$$

Par conséquent, le coefficient $a_0$ a comme expression

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt.$$

## 2. Somme partielle

Prenons la fonction $f(t)$ définie par

$$f(t) = \begin{cases} 1 & \text{si } 0 \leq x < \pi \\ -1 & \text{si } \pi \leq x < 2\pi \end{cases}$$

et étendue de manière $2\pi$-périodique sur $\mathbb{R}$, c'est-à-dire que

$$f(x + 2\pi n) = f(x) \quad (n \in \mathbb{Z},\, x \in [0, 2\pi)).$$

Cette fonction est illustrée dans la Figure 1. Ce qu'on veut calculer, ce sont les coefficients de Fourier. Vous pouvez vous convaincre (ou WolframAlpha peut vous convaincre) que les coefficients de Fourier de $f$ sont

$$a_0 = a_1 = a_2 = \cdots = 0 \text{ et } b_k = \frac{2}{n\pi}(1 - (-1)^n).$$

La série de Fourier associée à la fonction $f$ est donc

$$\sum_{n\geq 1} \frac{2}{n\pi}(1 - (-1)^n)\sin(nt).$$

Pouvons-nous utiliser la série de Fourier afin d'évaluer les valeurs de la fonction $f$? Pouvons-nous approximer convenablement les valeurs de notre fonction $f$ en exploitant une partie de sa série de Fourier?

Pour répondre à cette question, nous devons savoir comment on évalue une série!

*Définition* 1. Une série $\sum_{n\geq 0} x_n$ est dite convergente (où $(x_n)$ est une suite de nombres réels) si la limite de ses sommes partielles

$$s_n := x_0 + x_1 + x_2 + \cdots + x_n$$

existe, c'est-à-dire il existe un nombre réelle $s$ tel que

$$\lim_{n\to\infty} s_n = s.$$

Le nombre $s$ est appelé la *somme* de la série $\sum_{n\geq 0} x_n$ (si ce nombre existe!) et on pose

$$\sum_{n\geq 0} x_n = s.$$

Si aucun nombre $s$ n'existe tel que $\lim_{n\to\infty} s_n = s$, alors on dit que la série $\sum_{n\geq 0} x_n$ diverge. Notre question initiale est donc équivalent à la suivante : est-ce que la limite de la suite $(s_n(t))$ existe en tout point $t \in [0, 2\pi)$ où

$$s_n(t) := \sum_{k=1}^{n} \frac{2}{k\pi}(1 - (-1)^k)\sin(kt) \tag{5}$$

et peuvent-elles servir à approcher notre fonction $f$ convenablement (c'est-à-dire avec une petite erreur) ?

## 3. Le phénomène de Gibbs

Malheureusement, les sommes partielles d'une série de Fourier peuvent très mal approcher les valeurs d'une fonction! Sur la Figure 2, la fonction $f(t)$ est tracée en mauve et les graphes des sommes partielles $s_n(t)$, pour $n = 2, 4, 16$ et $64$ sont aussi tracés.

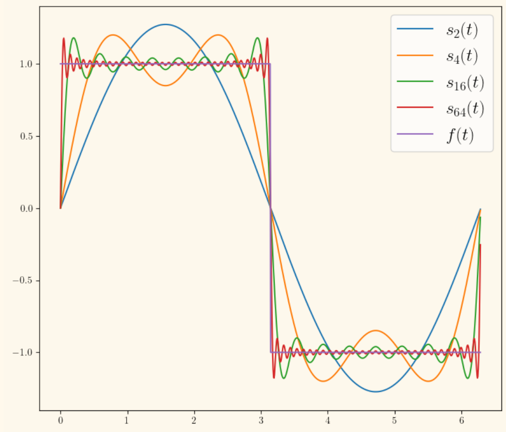



Figure 2: Sommes partielles de la série de Fourier

Notons les oscillations près du point $t = \pi$ (où il y a la cassure). La Figure 3 illustre bien ce phénomène (il s'agit d'un zoom autour du point $t = \pi$ du graphe de la Figure 2).

On remarque que la distante entre la plus grande bosse des sommes partielles $s_2(t)$, $s_4(t)$, $s_{16}(t)$ et $s_{64}(t)$ et la fonction $f(t)$ est plus ou moins constante et ne diminue jamais en dessous d'environ 0.2. Même si on augmentait la valeur de $n$ (c'est-à-dire le nombre de termes dans la somme partielle) et donc la précision, les oscillations vont toujours demeurer présentes.

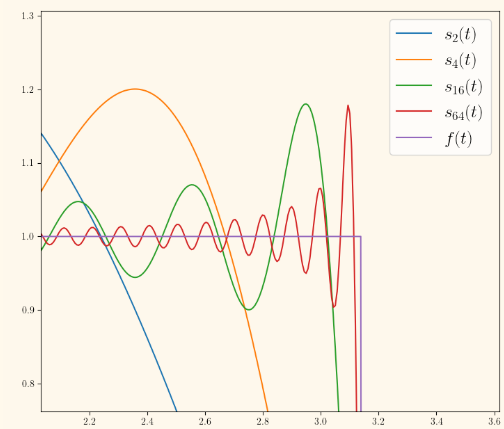



Figure 3: Zoom autour de la région $t = \pi$

En fait, il y a une borne inférieure à la grandeur des oscillations. Sur le graphe, on peut calculer à peu près que cette borne inférieure est 0.2, mais il est possible d'avoir une borne précise... C'est le phénomène de Gibbs!

*Théorème* 1 (Phénomène de Gibbs). Avec $t_n = \frac{n-1}{n}\pi$ pour $n \geq 1$, la différence entre les sommes partielles et la valeur 1 est bornée inférieurement de la manière suivante:

$$|s_n(t_n) - 1| > \frac{2}{\pi}\int_0^{\pi} \frac{\sin\psi}{\psi}\, d\psi - 1 \approx 0.17897974.$$

Ceci indique que, quelque soit le nombre $n$, il y aura toujours un nombre $t_n = \frac{n-1}{n}\pi$ telle que la quantité $|s_n(t_n)|$ a au moins une erreur de 0.17897974... sur la vraie valeur de la fonction $f(t)$. Ceci représente une erreur de près de... 17%!! On est mal barré si on veut une valeur de notre fonction autour des points $t_n$...

### 4. Moyennes de Cesàro

Néanmoins, il est possible d'atténuer ce petit problème en utilisant les moyennes de Cesàro d'une suite! L'argumentaire sera essentiellement basé sur les graphiques (pardonnez-moi...).

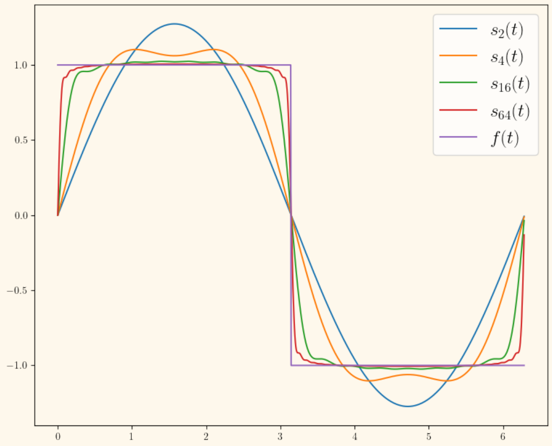



Figure 4: Graphes des moyennes de Cesàro

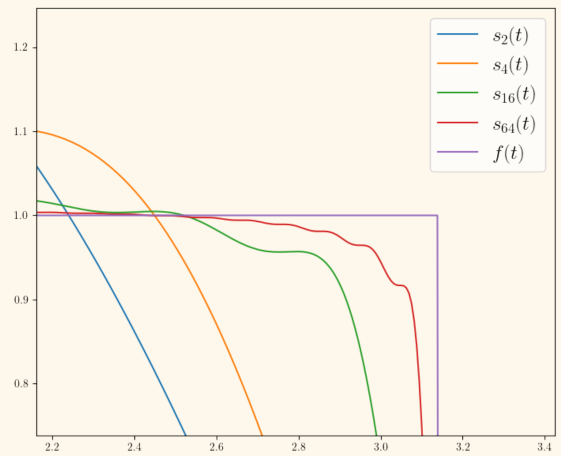



Figure 5: Zoom autour de $t = \pi$

*Définition* 2. Soit $(x_n)$ une suite de nombres réels. Les *moyennes de Cesàro de* $(x_n)$ sont définies par l'expression

$$\sigma_n := \frac{x_0 + x_1 + x_2 + \cdots + x_n}{n+1}.$$

Les moyennes de Cesàro sont appliquées à la suite des sommes partielles $(s_n(t))$ associée à la série de Fourier de notre exemple. Ainsi, on définit

$$\sigma_n(t) = \frac{s_0(t) + s_1(t) + s_2(t) + \cdots + s_n(t)}{n+1}$$

où $n \geq 0$ est un entier. On trace maintenant les graphes des fonctions $\sigma_2(t)$, $\sigma_4(t)$, $\sigma_{16}(t)$ et $\sigma_{64}(t)$. Ces graphes sont illustrés dans la Figure 4.

On observe une bonne amélioration puisque les oscillations drastiques autour du point $t = \pi$ ont complètement disparu! En faisant un petit zoom près de $t = \pi$ comme dans la Figure 5, on peut voir qu'il y a encore une erreur entre la valeur de la fonction près de $t = \pi$ d'au moins 0.1 près de la valeur de $t = \pi$. Ceci est causé par un autre phénomène, dû au fait que notre fonction est linéaire par morceaux.

*Théorème* 2. Si une fonction $f(t)$ est linéaire par morceaux, alors

$$\lim_{n\to\infty} s_n(t) = \frac{f(t^+) + f(t^-)}{2}$$

pour tout $t \in \mathbb{R}$.

Les nombres $f(t^+)$ et $f(t^-)$ sont les limites à droite et à gauche de la fonction $f(t)$ respectivement. Ce dernier résultat est un cas particulier du théorème de Dirichlet sur la convergence des séries de Fourier de fonctions différentiables par morceaux.

Dans notre exemple, la function $f(t)$ est linéaire par morceaux. De plus, lorsque $t \in (0, \pi)$, nous avons $f(t^-) = f(t^+) = 1$ et par le théorème de Dirichlet

$$\lim_{n\to\infty} s_n(t) = 1.$$

Par le lemme de Cesàro (un excellent exercise d'analyse réelle!), on déduit que

$$\lim_{n\to\infty} \sigma_n(t) = 1.$$

En suivant la même démarche, on a aussi que $\sigma_n(t) \to -1$, pour $t \in (\pi, 2\pi)$. De plus, comme $s_n(\pi) = 0$ pour tout $n$, nous avons que $\sigma_n(\pi) = 0$ pour tout $n$. Les fonctions $s_n(t)$ sont aussi lisses et donc les moyennes de Cesàro sont aussi des fonctions lisses.

Ces particularités expliquent brièvement l'écart présent entre $\sigma_{64}(t)$ et $f(t)$ près de $t = \pi$ : les courbes des $\sigma_n(t)$ sont lisses, doivent approcher les valeurs 1 et $-1$ autour the $\pi$, mais doivent passer en 0 lorsque $t = \pi$.

Bref, au moins, l'approximation par les moyennes de Cesàro nous a débarassé des grandes oscillations pertubantes provenant du phénomène de Gibbs! ◀

Pierre-Olivier Parisé

Ma passion pour le surf est tellement intense que même en visitant le Canada, je n'ai pas pu résister à l'envie de surfer dans ses eaux glaciales! ◀

## References


[1] Y. Katznelson. *An introduction to harmonic analysis*, $3^{\text{ième}}$ edition, Cambridge univerisity press, 314 pp. (2004).

[2] E. Stade. *Fourier Analysis*, Wiley, 488 pp. (2005).

# Experiencing CUMC 2023

By: Kanav Madhura (University of Toronto)

Kanav Madhura

This is all just abstract nonsense. What is the point? I don't know, I'm not a philosopher
- My set theory professor

One of the best parts of mathematics is how expansive it is. There is no one person that can understand all of it; that is why it is so important to foster communication between prospective mathematicians. It is with this goal that the Canadian Undergraduate Mathematics Conference, which I had the pleasure of attending this year, is run.

The heart of the conference was the student talks. Every attendee was encouraged to give a short 20-minute talk on any topic of their choosing. There were usually about four talks occurring simultaneously, and it was up to every person to decide which one to attend. What seems like much too little time was often more than enough to pique interest and lead to questions and discussions - not unlike those after a particularly engaging lecture. However, I believe that the people that got the most value out of these talks were the speakers themselves. Learning how to communicate ideas effectively and succinctly is a skill that is honed separately from studying and homework, so unfortunately it is not emphasised in a typical bachelor's degree in math. This conference gave people the opportunity to try their hand at it in a low-stakes environment, or improve if they already had experience.

It wasn't only students that gave talks, however. We had plenty of guest speakers, all of whom were professors from the University of Toronto, where the conference was taking place. From analysing Picasso through the lens of projective geometry to learning about how comathematicians turn cotheorems into ffee, we were treated to delightful expositions on a variety of subjects by experts in their fields. While the hour-long talks were more in line with what one would expect from a typical conference, the professors knew their audience and tried not to confound us too badly.

Of course, one of the big reasons to attend a conference is to be able to socialise and discuss mathematics with like-minded individuals. Every day after the main events of the conference there was an event planned by the amazing student-led Organising Committee. One night they rented out a bar for us and another we had a games/movie night, all with plenty of food. While there were inevitably discussions about math, it was a great chance to unwind after the onslaught of the day and get to know fellow attendees beyond their mathematical inclinations.

Although I won't be an undergraduate next year, I strongly recommend that, given the opportunity, every prospective mathematician attend CUMC 2024. I can only hope that conferences I attend in the future live up to this one. ◀

# Winter Puzzles

By: Yuliya Nesterova (Carleton University)

Did you know that the brain needs about 20 watts when thinking?* Warm up this winter with these puzzles!

## Puzzle 1

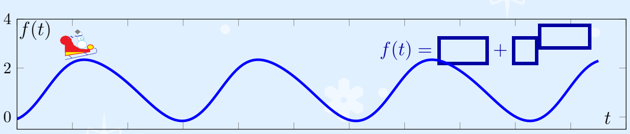
Each letter represents a unique digit! Even digits are light-coloured and odd digits are dark-coloured.

Each ❄ covers up a carried digit (i.e. carry-the-one, carry-the-two, or carry-the-three)

| | | | | | | |
|---|---|---|---|---|---|---|
| | | ❄ | | | ❄ | |
| | | | | | | A |
| | | | N | F | T | M: |
| | W | I | N | T | E | R |
| + | | | | F | U | N! |
| | W | A | R | M | I | I |

## Puzzle 2

Sintwo Cos, the magic elf who brings good math marks on December exams, wants to build a good roller coaster launcher for his magic sleigh. He got the blueprints from his neighbour, who only remembered some cool properties about the curve functions: can you work out the formula?




The first term and the second term's exponent are trig functions with period $2a$, where $a$ is the solution to $\frac{Area_{circle}}{radius^2_{circle}}$.

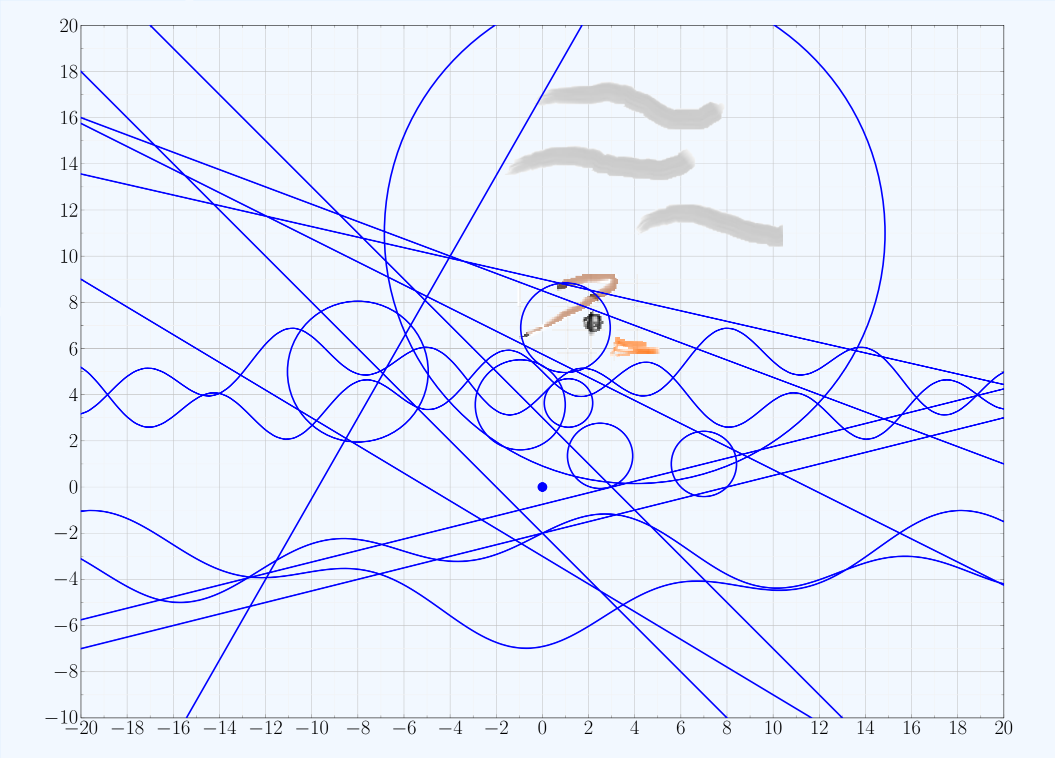
The base of the exponent in the second term is the length of the interval of both trig functions' ranges. It is also the only even prime.

The exponent had an intercept at $(0, 1)$ and the first term ran through the origin.

* Balasubramanian, Vijay. "Brain Power." *Proceedings of the National Academy of Sciences - PNAS*, vol. 118, no. 32, 2021

## Puzzle 3

A picture is worth a thousand functions! Trace out the functions below at the domains indicated and see what picture lurks in the scribbles




$$\text{dark blue}\begin{cases} \dfrac{x}{4}-2 & x\in(-5,8) \\ 3-x & x\in(3,4) \\ -2-x & x\in(-1,0) \\ (x-7)^2+(y-1)^2=2 & \text{major arc from } (6,2) \text{ to } (8,0) \end{cases}$$

$$\text{red}\begin{cases} \dfrac{x}{4}-\dfrac{3}{4} & x\in(-2.6,3) \\ (x-2.5)^2+(y-1.35)^2=2 & \text{major arc between } (1.3,0.6) \text{ and } (3,0) \\ \dfrac{-3}{5}x-3 & x\in(-9,-3) \\ (x-4)^2+(y-11)^2=117.8 & \text{minor arc from} (-5,5) \text{ to } (1.25,0.5) \\ (x+8)^2+(y-5)^2=9.3 & \text{major arc from} (-8,1.95) \text{ to } (-5,5) \end{cases}$$

$$\text{purple}\left\{ \quad 4-\sin(x)+\cos\left(\frac{x}{4}\right) \qquad x\in(-18,-1) \right.$$

$$\text{light blue}\begin{cases} (x-1)^2+(y-6.9)^2=3.77 & x\in(-\infty,\infty) \\ (x+0.96)^2+(y-3.56)^2=3.82 & x\in(-\infty,\infty) \\ (x-1.13)^2+(y-3.64)^2=1.1 & x\in(-\infty,\infty) \end{cases}$$

$$\text{brown}\begin{cases} 9-\dfrac{57}{250}x & x\in(-4,1) \\ \dfrac{23}{4}-\dfrac{1}{2}x & x\in(-5,-1) \\ 1.75x+17 & x\in(-5,-4) \end{cases}$$

Yuliya Nesterova

I use an Addiator to make some of my puzzles: I love to collect old analog instruments! ◄

# Exploring the Saturn Ring Defect for Two Colloidal Particles

By: Spencer Locke and Hayley Monson. *Supervisors: Dr. Lia Bronsard and Dr. Dominik Stantejsky* (McMaster University)

The study of nematic liquid crystals and the defects that arise due to various energy configurations has been of great interest to both mathematicians and physicists alike. These "defects" refer to singularities in the *order parameter*, which is a quantity that measures the level of order in the arrangement of molecules within a liquid crystal solution. In their 2016 paper, Alama, Bronsard & Lamy used the Landau-de Gennes model for nematic liquid crystals with a single spherical colloid particle to show that a ring defect forms when the particle is sufficiently small [1]. For this reason, they dubbed it the *Saturn ring defect*! To explicitly find the ring, they showed, using variational principles on the Landau-de Gennes energy functional, that one must solve for a quantity known as the $Q$-tensor, which will minimize this functional. In this case, the tensor will be a symmetric and traceless solution to the Laplace equation with Dirichlet boundary conditions on the surface of the ball, and with uniform alignment far away. It was also shown successfully that the defect must occur at every point in space where there is a crossing of the dominant eigenvalues from this $Q$-tensor.

The discovery of this ring defect for one colloid inspired the investigation of how the defects, if any exist, appear in the presence of two particles. To do this, we considered two colloids in a liquid crystal solution, solving for the minimizing $Q$ tensor in order to find the crossing points of its eigenvalues. Just as in the one ball case, $Q$ is found as the solution to the analogous Dirichlet boundary value problem, which is given explicitly as:

$$\begin{cases} -\Delta Q = \mathbf{0} \\ Q|_{\partial B_1} = \nu_1 \otimes \nu_1 - \frac{Id}{3} \\ Q|_{\partial B_2} = \nu_2 \otimes \nu_2 - \frac{Id}{3} \\ Q \to Q_z \equiv e_3 \otimes e_3 - \frac{Id}{3} \text{ as } \|(x,y,z)\| \to \infty \end{cases}$$

where $\partial B_1$ and $\partial B_2$ denote the boundaries of our two colloidal particles, $\nu_1$ and $\nu_2$ are the unit normal vectors to their surface, and $Q$ itself is a rank 2 tensor represented as a $3 \times 3$ symmetric and traceless matrix. In this case, the uniform alignment is taken perpendicular to the $x$-axis, which is where we'll position the colloids.

Using linearity of the Laplacian, we obtained that the superposition of solutions given by $Q = Q^1 + Q^2 + (1 - g_1 - g_2)Q_x$ is a solution, where $Q^1$ and $Q^2$ are also rank 2 tensors, and where, for $i = 1, 2$, $Q^i$ solves the Laplace BVP with Dirichlet boundary conditions on $\partial B_i$, and vanishing conditions on the other ball. In addition, $g_i$ is the real-valued function satisfying the Laplace BVP which is 1 on $\partial B_i$ and vanishing on the other ball. In both cases, we also take $Q^i$ and $g_i$ tending to zero far away in order to satisfy the original boundary conditions.

To treat both spherical boundary conditions separately, we converted our domain to *bispherical coordinates*, which is a system of orthogonal curvilinear coordinates $(\eta, \chi, \theta)$. When constructed along the $x$-axis, the system is described by the intersection of two families of circles, and is given by the following change of variables:

$$x = \frac{a \sinh(\eta)}{\cosh(\eta) - \cos(\chi)}, \; y = \frac{a \sin(\chi)\sin(\theta)}{\cosh(\eta) - \cos(\chi)}, \; z = \frac{a \sin(\chi)\cos(\theta)}{\cosh(\eta) - \cos(\chi)}$$

where the foci of both families of circles are located at $x = \pm a$, with $0 < a < \infty$, and is determined by the location and size of the colloids. In this coordinate system, the variable $\eta$ fixes the location of the spheres on the $x$-axis, while $\theta$ fixes cross-sections of the spheres in the planes described by $\tan\theta = \frac{y}{x}$, and $\chi$ fixes points along these cross-sections. Thus, if we place our two balls on the $x$-axis, we may uniquely identify values $\eta_1$ and $\eta_2$ which are such that $Q|_{\partial B_1} = Q(\eta_1, \chi, \theta)$ and $Q|_{\partial B_2} = Q(\eta_2, \chi, \theta)$, hence allowing us to easily restrict to the boundaries.

Crucially, it can be shown that the Laplace equation is actually partially separable in bispherical coordinates. Taking the resulting general solution after separation of variables, we were able to apply the Laplacian component wise to both $Q^1$ and $Q^2$ to solve for all their components, and also to $g_1$ and $g_2$. This allowed us to obtain a closed form solution for our $Q$-tensor by using the linear decomposition previously mentioned.

Of course, the goal here was to discover the Saturn ring defects for the two colloid case, so by obtaining the positions of where the dominant eigenvalues of $Q$ cross in space, we can plot these defects! Due to the complicated nature of the $Q$-tensor, we obtained a full picture by running numerical simulations of $Q$, looping through various values of $\eta$, $\chi$ and $\theta$ in our solution, and determining whether there was a crossing of the two largest eigenvalues at each point in space. Fig. 1 demonstrates the results of these simulations, when taking two colloids with both radii equal to 1 and whose centers are separated by a distance of 2.5:

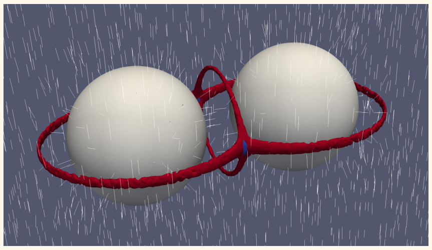


Figure 1: Two colloids submerged in a nematic liquid crystal. The red rings demonstrate the Saturn ring defects, while the white lines indicate the vectorial direction of $Q$ at that point in space.

As a result of this simulation, we observe evidence for the existence of the Saturn ring defects for two colloids! Not only that, though, but there is an extension from the one colloid case; there, we saw there was only one ring that formed, wrapping around the colloid. In this case, we see a separate ring wrapping around each colloid, but which also intersects with a ring that is located identically in the $yz$-plane, thus lying half-way between both colloids. These are just some of the many properties of the Saturn ring defect yet to be explored for the two colloid case! ◀

## References

[1] Alama, S., Bronsard, L. & Lamy, X. "Minimizers of the Landau–de Gennes Energy Around a Spherical Colloid Particle." *Arch Rational Mech Anal* **222**, 427–450 (2016).

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# What's it like working as a mathematician in the public sector? - An interview with Dr. Megan Dewar

By: Courtney Allen (University of Guelph)

Dr. Megan Dewar is the head of the Tutte Institute for Mathematics and Computing (TIMC). I sat down with Dr. Dewar to discuss her experiences working in the public sector, the pros and cons, and what students can do to bolster their chances of building a career in government.

*This interview was conducted in two parts. First Dr. Dewar responded to written questions via email, and then we met virtually to discuss her answers. The following is a combination of both her written and verbal responses and has therefore been edited for clarity and length.*

**First off, do you want to tell us a bit about TIMC?**

TIMC is a government research institute that is part of the Communications Security Establishment (CSE) – the Canadian government's national cryptologic agency, providing the Government of Canada with information technology security and foreign signals intelligence. TIMC's mission is to deliver research results with an impact on the most important scientific challenges facing the Canadian and Five Eyes security and intelligence communities. The Five Eyes refers to five countries, Canada, the US, the UK, Australia, and New Zealand. They're a group of affiliated nations that work together in cyber security to secure their communications and defend against threats. TIMC is part of the 5-Eyes mathematics research community which is similar to the academic research community, but whose focus is on classified problems.

TIMC is named after William Tutte, one of the fathers of modern graph theory and a founder of the Combinatorics and Optimization Department at the University of Waterloo. He also worked as a code breaker at Bletchley Park during World War II. He's less well known than Turing but did a similar kind of cryptoanalysis that helped the Allies win World War II.

*You can learn more about the institute at: https://www.cse-cst.gc.ca/en/mission/research-cse/tutte-institute-mathematics-computing.*

**What does your work at TIMC entail and how are you using your math degree in your current job?**

As head of the institute, I have five main tasks: to distil the priorities of CSE into research requirements; to set TIMC research direction to meet CSE's needs; to encourage an atmosphere of collaboration and innovation; to support my staff to achieve their objectives; and to engage with other areas of CSE, other government departments and international partners, academia, and industry. Occasionally I collaborate with colleagues on research questions, but mostly I champion and support the research work of others.

My field of academic study is combinatorics, specifically graph theory and design theory. While I no longer use my deep knowledge of mathematics against challenging research problems, I do use that knowledge daily to understand what my team is working on, where there are connections to be made between researchers or research groups, and to give advice and guidance to other areas of CSE on technical problems. There's a basic logic application that you learn in math that makes you ask questions. For someone who's leading a research institute that is focused on mathematics, it is important for me to be able to understand how complex that mathematics is. Being able to understand that complexity also helps when communicating with people who aren't technical experts. Because I'm not in an academic setting, I'm often speaking to non-experts and trying to explain the impact of our work and why the mathematics matters.

**Communication is a really underrated part of a math degree.**

It's a thing we try to teach when we have co-op students in. I think that universities have gotten better at teaching people how to give presentations and communicate, but often it's still framed for an academic audience. So when we have students we try to work on that piece where you're trying to communicate with someone who doesn't understand the math and doesn't really need to understand the math, but you want to convey enough about how hard it is or what the key points are, and have them come away understanding why you did what you did and why that route was taken vs another.

**How did you become interested in your current field?**

Honestly, I sort of fell into it. My partner was working at CSE when I finished my Masters degree, so I moved from Halifax to Ottawa to be with him. After a brief time working as a Defence Scientist at the Department of National Defence, I took a job in one of the math research groups at CSE. And not long after that I received funding from NSERC to support my doctoral studies so I took three years of education leave to do that.

**You're also an Adjunct Professor at Carleton University, how does that tie into your work at TIMC?**

My full-time job is with TIMC, but some of that role is to engage with academia and the broader Canadian mathematics community which I do mostly through activities with the CMS. At TIMC we have a strong focus on foundational research. While much of our research work is classified, we feel it's important for researchers at TIMC to maintain their presence in the academic community. This allows people to advance their skills, build an external research profile – which has the side benefit of helping us to recruit – and continue to do some research that is simply for the love of mathematics. Having an adjunct position at Carleton means I can co-supervise students, teach a course, and generally keep a connection with my academic colleagues.

**What does an entry level government position look like for someone who has an undergrad degree vs a master's or a PhD?**

It's a little hard to say because most of my experience is at CSE, more specifically at TIMC, so I'm on shaky ground talking about the public sector in general. What I would say is that with an undergrad degree in mathematics you are probably going to find a job that's not really mathematics based but more coding based. With a master's you're more likely to be bridging that gap there. You could be advising on policy from a very technical standpoint, or you could be doing some mathematics research. With a PhD, you've invested so much in that field, you're more likely to be doing work similar to academia but in the government setting.

That said, I do have colleagues who have advanced degrees, and they're not using those degrees in their public sector roles. I think TIMC is an oddity – where you may continue doing very deep mathematical research and where you have the opportunity to retain ties with the academic community – that I think is not that common generally in the public sector.

Megan Dewar

One quote (from Aristotle) that I come back to time and again is not one I necessarily truly believe/agree with, but which I find fascinating for its emphasis on the third "feature" of mathematics: *The mathematical sciences particularly exhibit order, symmetry, and limitation, and these are the greatest forms of the beautiful.* I do think that "limitation" forces questions and creativity!

**How far ahead should students think about applying, especially for higher security jobs?**

They should start as early as they possibly think they're interested. While CSE hires at all security levels, the majority of our work requires a Top Secret clearance, which means that our hiring process is more demanding than most. But that investment of time is worthwhile – the work of CSE is extremely interesting and CSE has a great culture of inclusivity and innovation. In particular, CSE has a great co-op program which hires students from undergrad, masters, and PhD. You want to apply at least 9 months ahead of the term you're applying for. The applications for people who want to come next summer closed in early October. I always say to people, if you're thinking about it at all, and you're not sure where you're going to end up in a year or two, just put in your application, because it could take a while. It is a requirement for working at CSE that you be a Canadian citizen, and you need Top Secret clearance so there are a lot of processes to work though.

I do recommend that students come and see what it's like. For CSE, it is one of the main ways that we do our recruiting. You come for a term, you see what it's like, you may even come back with another group later because now you've got that security clearance, and if you perform well and you like it here, it's very likely that you'll have a job at the end of that. It's a great transition if you want a job at the end of your degree and you're interested in working for the government. Starting with a co-op program and working through it is a really effective route to full-time employment at CSE, and I'd expect that's the case in many other federal government departments too.

**Many mathematics students consider an academic job by default, why might a student consider a government job instead?**

I think considering options outside academia is a good idea. Lots of research and innovation happens outside academia – shameless plug for the session Kseniya Garaschuk and I are organizing at the upcoming CMS Winter Meeting on *Research in the Public Sector*!

I had a great supervisor [in my PhD], and the problem I was working on was really interesting, but I had a lot of concerns about going out into the academic world and continuing to have to find interesting problems and develop collaborations. What was nice about a government job was that you had the freedom to work on things that you had expertise in, but it was constrained to certain problems. Also, your colleagues are quite close to you. You're all working towards a similar set of goals and you have the support of your colleagues.

The government has lots of programs to retain people. They really want to support their employees, so there are lots of different programs including education leave of various sorts. They will make sure your job, or a similar job, is available to you when you come back, and there are ways in which you can get paid for some of that time off. Then there are other types of leave that you can take, and they also provide lots of opportunities for on-the-job training. In the federal government, if you want to move into management you need to be bilingual, so they provide lots of training for people who are interested in developing second language skills too.

**Do you get more of a work-life balance in a government job?**

I think globally speaking there is more of a work-life balance. There's a standard 37.5 hour work week. I do think the government balance is much better than the academic balance. With industry, it depends where you're working, especially if you're with a start-up, that can be kind of all-consuming as well.

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**If a student is interested in a job with the government, what can they do to improve their chances?**

When I give career-type talks I tend to highlight the following ”tips”:

- Develop implementation skills [ie. coding skills]. Especially if you have only an undergrad in mathematics, you're more likely to be working in that implementation space. Having the ability to pick some of that up is really helpful.
- Consider depth and breadth. For those people with a PhD, you've gone down into pretty narrow subject matter. If you're thinking about other jobs than continuing in academia, it can be helpful to have some knowledge outside that and to think about how the knowledge you have applies to other fields.
- Don't underestimate the importance of "soft" skills, especially communication. Collaboration is also key. Being able to work in a group or lead a meeting. When we're assessing people's capacities to do a job, we've got this list of competencies, and some of them are technical, but a huge number of them are behavioural and leadership. People tend to undersell, for example, writing a column for Notes from the Margin, or participating on a student committee. When I'm hiring, those are things that I make note of. Did this person focus only on their academics, or did they do other things? A well-rounded person can be a much better fit in government or industry, where you're working in a team or group, than someone who has very niche expertise and is focused very inwardly.
- Explore options for co-op placements, internships, work terms, and so on.
- Have patience and persistence!

**Finally, where can students find more information about getting a government job?**

CSE has it's own web page (https://www.cse-cst.gc.ca/en) and you can find both professional and student opportunities off that page. There's a general Government of Canada webpage for that too (https://www.canada.ca/en/services/jobs/opportunities/government.html). There are a few other separate employers, like CSIS and Canada Post, where you'd have to go through their individual organization, but for bigger government organizations that's the main portal. ◀


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