Image credit: Philippe-André Luneau

Notes from the Margin

Volume XIV · 2023

A First Result in Complex Approximation

By: Alexander Kroitor (University of Waterloo)

Select an interval $[a, b]$ in $\mathbb{R}$ and $\epsilon > 0$. A well-known theorem by Weierstrass [1], proven in 1885, states that for any function $f(x)$ from $\mathbb{R}$ to $\mathbb{R}$ continuous at each point in $[a, b]$ there exists a polynomial with real coefficients $P(x)$ (depending on $[a, b]$, ϵ, and f) that uniformly approximates f on the interval $[a, b]$ up to error ϵ. That is, $|f(x) - P(x)| < \epsilon$ for all $x \in [a, b]$.

It is natural to look at this result and think of how it can be generalized. There are multiple directions to try, and it turns out that moving to complex functions rather than real functions yields some beautiful results. This motivates the rich topic of **complex approximation**, the field of approximating functions from $\mathbb{C}$ to $\mathbb{C}$. In order to delve in properly, we recall from complex analysis that a function f is **analytic** at a point z_0 if f can be expanded as a power series centered at z_0 (we can think of this as f being "nice" at z_0).

In 1951 Mergelyan proved the following powerful result [2]. First, select a compact set K in $\mathbb{C}$ such that the complement of K is connected, that is we cannot write the complement of K as a disjoint union of two other non-empty open sets. Then given a function $f(z)$ from $\mathbb{C}$ to $\mathbb{C}$ that is continuous at each point in K, and analytic at each point in the interior of K, there exists a polynomial with complex coefficients $P(z)$ (once again depending on K, ϵ, and f) that uniformly approximates f on K up to error ϵ.

The requirement that f be analytic may look restrictive, but the fact that the analyticity requirement is only for the interior of K renders this quite benign. We can achieve powerful results by constraining ourselves to sets that have no interior. Recall that given a complex number z in $\mathbb{C}$ we can split it into its real and imaginary parts as $z = \operatorname{Re}(z) + i \operatorname{Im}(z)$, where $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ are real numbers. We can do the same to functions: given a function $f(z)$ we can split it into its real and complex parts $f(z) = \operatorname{Re} f(z) + i \operatorname{Im} f(z)$ where $\operatorname{Re} f$ and $\operatorname{Im} f$ are functions from $\mathbb{C}$ to $\mathbb{R}$. It is straightforward to see that if P is a polynomial with complex coefficients, then $\operatorname{Re} P$ is also a polynomial with real coefficients.

Mergelyan's Theorem has Weierstrass's Theorem as a corollary. Pick an interval $[a, b]$ in $\mathbb{R}$ and a function $f(x)$ that is continuous on $[a, b]$, and define

$$K := \{z \in \mathbb{C} \mid \operatorname{Re}(z) \in [a, b], \operatorname{Im}(z) = 0\},$$
$$g(z) := f(\operatorname{Re}(z)) \in \mathbb{R}.$$

Since g is continuous, and K has no interior, Mergelyan's theorem gives us a complex polynomial $P(z)$ such that $|g(z) - P(z)| < \epsilon$ on K. [Continued on page 2]

Alexander Kroitor

When not grading and giving tutorials, I occasionally squeeze in time for math.

Preamble

By: Courtney Allen (University of Guelph)

Courtney Allen
Acting Editor-In-Chief

What does it take to be a mathematician? [...] It does not take brilliance, but love of a great game!

- Karen Keskulla Uhlenbeck

It's a common question heard in math classes the world over: "Why do I have to show my work?" It's a good question: if the answer's right, then why does it matter how I got it? While "showing you work" often seems like busy-work, if you've taken even one course in so-called "advanced" mathematics, you know the truth. The *only* thing that matters is your work.

If you've known me in real life for a sufficiently long period of time, you've probably heard me say, in an exasperated tone of voice, "Math is about communication!" because, well... it is. Nobody cares that the answer is 42 if you can't explain how you got there. Mathematics is a tool that we use to describe the world around us.

It's for that reason that I'm so happy to be bringing back *Notes from the Margin* after its extended hiatus. Providing an outlet for mathematics students to communicate and discuss mathematics is integral to a well-rounded mathematical education, and it's a privilege to have collaborated with such brilliant contributors on this edition.

In this issue we take a dive into complex analysis, linear algebra, and take a look at an important figure in the history of mathematics. We also celebrate the work of the winners of the Summer 2022 and Winter 2022 AARMS-CMS Student Poster Session.

If you want to see your work in *The Margin*, or if you have questions or comments about the articles in this issue, contact the editor at **student-editor@cms.math.ca**.

[Continued from cover page]

Since $|a+ib| \geq |a|$, we have that, for z in K,

$$\begin{aligned} \epsilon &> |g(z) - P(z)| \\ &= |f(\mathrm{Re}(z)) - \mathrm{Re}\,P(z) - i\,\mathrm{Im}\,P(z)| \\ &= |f(z) - \mathrm{Re}\,P(z) - i\,\mathrm{Im}\,P(z)| \\ &\geq |f(z) - \mathrm{Re}\,P(z)|, \end{aligned}$$

and so on K (and thus on our interval $[a,b]$) we have found a real polynomial $\mathrm{Re}\,P$ that uniformly approximates f up to error ϵ.

This is only a small result in complex approximation, and still has the requirement that the complement of K is connected. Removing this requirement leads to approximations with rational functions instead of polynomials. Further generalization lead to an elegant proof of the beautiful **Birkhoff's Universality Theorem**, but that would be a theorem for another article.

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In This Issue:

L'enveloppe spectrale des matrices bistochastiques:

Une étude de cas du comportement étrange des petits nombres

By: Ludovick Bouthat (Université Laval)

Une matrice carrée est dite *stochastique* si elle est non-négative et si la somme des coefficients de chaque ligne est égale à 1. De même, une matrice carrée est dite *bistochastique* si elle est non-négative et si la somme des coefficients de chaque ligne et de chaque colonne est égale à 1. De manière équivalente, une matrice est bistochastique si la matrice et sa transposée sont toutes deux stochastiques. Ici, nous dénoterons l'ensemble des matrices $n \times n$ bistochastiques par $\mathcal{D}_n$.

En 1938, lors d'une conférence sur les chaînes de Markov organisée sous l'égide de la Société mathématique de Moscou, le célèbre mathématicien Andreï Kolmogorov définit Ω_n comme l'ensemble de toutes les valeurs propres de toutes les matrices stochastiques $n \times n$ et pose le problème de la détermination de cette région. Treize ans plus tard, en 1951, F. Karpelevič [1] obtenu finalement une description complète de Ω_n pour tout $n \geq 1$.

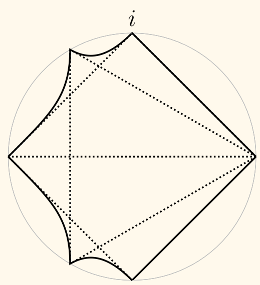

Figure 1: Ω_4

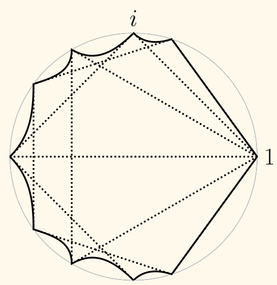

Figure 2: Ω_5

Les régions Ω_4 et Ω_5 et les polygones réguliers capturant les extrémités.

Une question analogue, proposée par L. Mirsky en 1963 [2], consiste à déterminer la région ω_n de toutes les valeurs propres de toutes les matrices $n \times n$ bistochastiques, c'est-à-dire $\omega_n := \{\lambda \in \mathbb{C} : \lambda \in \sigma(D), D \in \mathcal{D}_n\}$. Puisque toute matrice bistochastique est également stochastique, nous avons clairement $\omega_n \subseteq \Omega_n$. De plus, posons $\Pi_n = \mathrm{Conv}\{e^{2\pi i/n}, e^{2\times 2\pi i/n}, \ldots, e^{n\times 2\pi i/n}\}$, soit l'enveloppe convexe fermée des n^{e} racines de l'unité, qui est le n-gone régulier dans le disque ancré au point 1. Nous avons le résultat suivant, dû à Perfect et Mirsky [2].

Théorème. $\Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_n \subseteq \omega_n$.

Avec des méthodes relativement simples, on peut montrer que

$$\omega_2 = \Pi_1 \cup \Pi_2 = [-1, 1] \qquad \text{et} \qquad \omega_3 = \Pi_1 \cup \Pi_2 \cup \Pi_3.$$

Conjecture (Perfect–Mirsky)**.** $\omega_n = \bigcup_{k=1}^{n} \Pi_k$.

En 2006, Mashreghi et Rivard [3] ont identifié la matrice bistochastique

$$S_t := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & t & 0 & 1-t \\ 0 & t & 1-t & 0 & 0 \\ 0 & 1-t & 0 & 0 & t \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (0 \leq t \leq 1)$$

et ont observé que pour au moins $t \in [0.49, 0.51]$, S_t admet une valeur propre en dehors de la région conjecturée $\bigcup_{k=1}^{5} \Pi_k$. Ainsi, en laissant varier t et en calculant la valeur propre exceptionnelle de S_t, on obtient une courbe qui se situe partiellement en dehors de la région de Perfect–Mirsky (en rouge dans Figures 3 and 4).

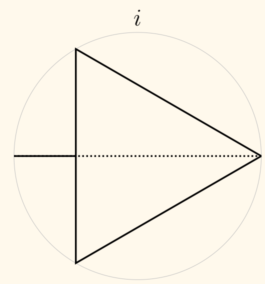

Figure 3: La région ω_3.

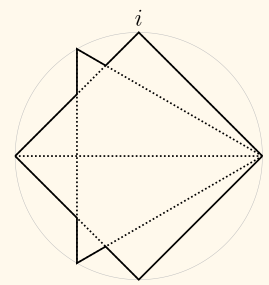

Figure 4: La région ω_4.

Cependant, l'histoire ne s'arrête pas là puisqu'en 2015, Levick, Pereira et Kribs [4] ont montré que la conjecture est également vraie pour $n = 4$.

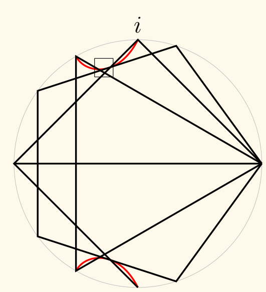

Figure 5: La région de Perfect–Mirksy $\bigcup_{k=1}^{5} \Pi_k$ et la courbe exceptionnelle de Mashreghi–Rivard.

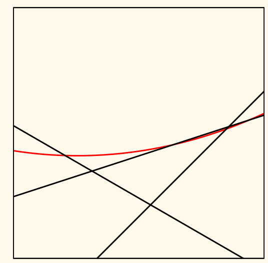

Figure 6: Gros plan sur la courbe exceptionnelle autour de $[-0.35, -0.2] \times [0.7, 0.85]$.

En ce qui concerne $n \geq 6$, le statut de la conjecture demeure en suspend. Cependant, Harlev, Johnson et Lim [5] ont récemment approché le problème numériquement. En se basant sur les travaux de Rivard et Mashreghi ainsi que sur leurs propres recherches, ils ont examiné les valeurs propres obtenues par des matrices bistochastiques qui peuvent être écrites comme une combinaison convexe d'au plus *deux* matrices de permutation pour $n \leq 11$ (comme c'est le cas pour S_t). Ils ont observés que pour chaque cas sauf $n = 5$, toutes les valeurs propres se trouvent dans $\bigcup_{k=1}^{n} \Pi_k$. Cela les a motivés à proposer la conjecture suivante.

Conjecture (Harlev–Johnson–Lim)**.** $\omega_n = \bigcup_{k=1}^{n} \Pi_k$ *pour* $n \geq 1$*,* ***sauf pour*** $\mathbf{n = 5}$**.**

Aussi troublante soit-elle, cette conjecture semble être la plus convaincante à ce jour, car elle est étayée par des calculs numériques et des propriétés algébriques de $\mathcal{D}_n$. Espérons qu'une nouvelle idée nous permettra de résoudre ce mystérieux problème. En attendant, nous concluons par la question ouverte suivante :

Question ouverte. *Quelle est la région* ω_n *pour* $n \geq 5$*? En particulier, peut-on caractériser la région* ω_5*?* ◀

Ludovick Bouthat

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.
- John von Neumann

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Ada Lovelace: The Woman Who Saw the Future

By: Courtney Allen (University of Guelph)

In 1843, an English translation of a French article was published in Taylor's Scientific Memoirs [1]. The article, originally written by Luigi Menabrea, outlined the workings of a hypothetical machine known as Babbage's Analytical Engine, an early programmable computer. It was wildly influential, not for the article itself, but for the notes made by the translator, a young woman by the name of Augusta Ada King, the Countess of Lovelace.

Ada Lovelace, as she is more commonly known, was the only legitimate child of Lord Byron, born in England on 10 December 1815. After her parents separation, her mother, Lady Byron, encouraged her to pursue the sciences. Since the sciences were then thought to be the province of men, Lovelace gained her education by reading textbooks and corresponding with some of the greatest mathematical minds of the time, one of whom was Charles Babbage [2].

Seeing her interest in his proposed Analytical Engine, Babbage encouraged Lovelace to read and translate the aforementioned paper by Luigi Menabrea [3]. But Lovelace did more than that, she saw the potential of the Engine in a way that no one had before. Her notes were longer than the article itself, and contain the first computer program, a table designed for the Engine that would compute the Bernoulli numbers [2]. More importantly, she saw the true power of the Analytical Engine, theorizing that it could be used to perform complicated tasks such as composing music [2]. Almost 100 years after her death from uterine cancer at age 36, her work continued to influence the inventors of the first modern computers, such as Alan Turing [3].

At a time when women's mathematical aptitude was often dismissed, Ada Lovelace saw the possibilities of a machine that did not yet even exist ◀

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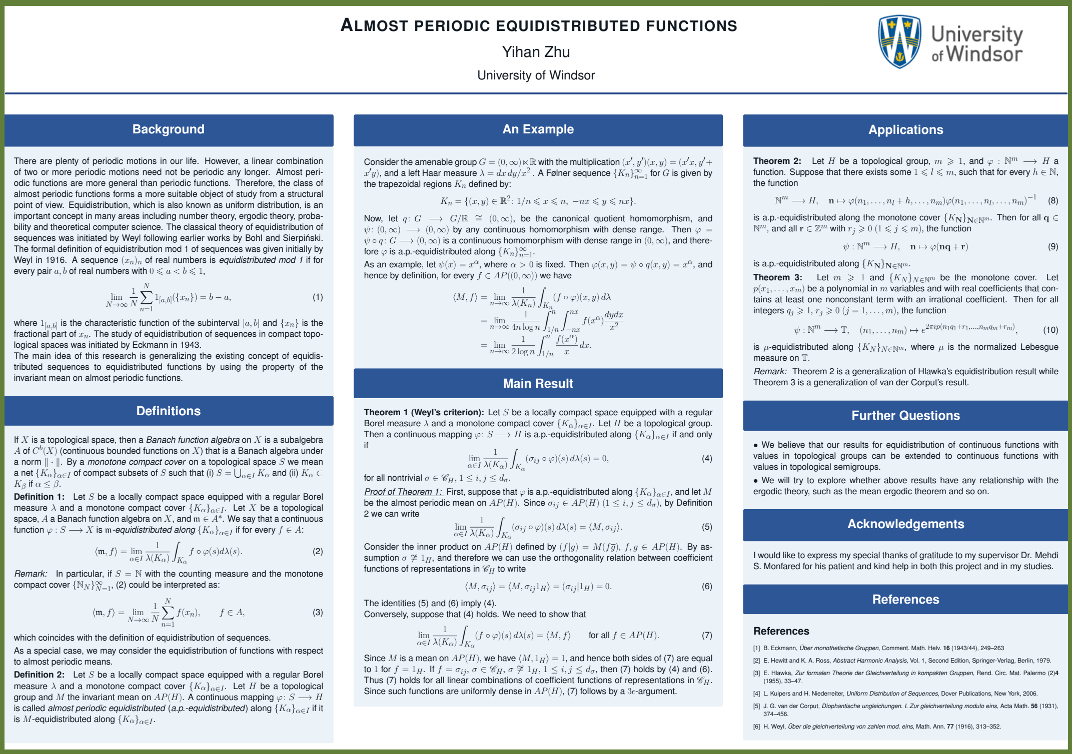
Student Poster Session Winners Summer 2022

Almost Periodic Equidistributed Functions

- Yihan Zhu (University of Windsor)


ALMOST PERIODIC EQUIDISTRIBUTED FUNCTIONS

Yihan Zhu

University of Windsor

Background

There are plenty of periodic motions in our life. However, a linear combination of two or more periodic motions need not be periodic any longer. Almost periodic functions are more general than periodic functions. Therefore, the class of almost periodic functions forms a more suitable object of study from a structural point of view. Equidistribution, which is also known as uniform distribution, is an important concept in many areas including number theory, ergodic theory, probability and theoretical computer science. The classical theory of equidistribution of sequences was initiated by Weyl following earlier works by Bohl and Sierpiński. The formal definition of equidistribution mod 1 of sequences was given initially by Weyl in 1916. A sequence $(x_n)_n$ of real numbers is *equidistributed mod 1* if for every pair a, b of real numbers with $0 \leqslant a < b \leqslant 1$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} 1_{[a,b]}(\{x_n\}) = b - a, \qquad (1)$$

where $1_{[a,b]}$ is the characteristic function of the subinterval $[a, b]$ and $\{x_n\}$ is the fractional part of x_n. The study of equidistribution of sequences in compact topological spaces was initiated by Eckmann in 1943.

The main idea of this research is generalizing the existing concept of equidistributed sequences to equidistributed functions by using the property of the invariant mean on almost periodic functions.

Definitions

If X is a topological space, then a *Banach function algebra* on X is a subalgebra A of $C^b(X)$ (continuous bounded functions on X) that is a Banach algebra under a norm $\|\cdot\|$. By a *monotone compact cover* on a topological space S we mean a net $\{K_\alpha\}_{\alpha\in I}$ of compact subsets of S such that (i) $S = \bigcup_{\alpha\in I} K_\alpha$ and (ii) $K_\alpha \subset K_\beta$ if $\alpha \le \beta$.

Definition 1: Let S be a locally compact space equipped with a regular Borel measure λ and a monotone compact cover $\{K_\alpha\}_{\alpha\in I}$. Let X be a topological space, A a Banach function algebra on X, and $\mathbf{m} \in A^*$. We say that a continuous function $\varphi : S \longrightarrow X$ is $\mathbf{m}$*-equidistributed along* $\{K_\alpha\}_{\alpha\in I}$ if for every $f \in A$:

$$\langle \mathbf{m}, f\rangle = \lim_{\alpha\in I} \frac{1}{\lambda(K_\alpha)}\int_{K_\alpha} f\circ\varphi(s)\,d\lambda(s). \qquad (2)$$

Remark: In particular, if $S = \mathbb{N}$ with the counting measure and the monotone compact cover $\{\mathbb{N}_N\}_{N=1}^{\infty}$, (2) could be interpreted as:

$$\langle \mathbf{m}, f\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n), \qquad f \in A, \qquad (3)$$

which coincides with the definition of equidistribution of sequences.

As a special case, we may consider the equidistribution of functions with respect to almost periodic means.

Definition 2: Let S be a locally compact space equipped with a regular Borel measure λ and a monotone compact cover $\{K_\alpha\}_{\alpha\in I}$. Let H be a topological group and M the invariant mean on $AP(H)$. A continuous mapping $\varphi : S \longrightarrow H$ is called *almost periodic equidistributed (a.p.-equidistributed)* along $\{K_\alpha\}_{\alpha\in I}$ if it is M-equidistributed along $\{K_\alpha\}_{\alpha\in I}$.

An Example

Consider the amenable group $G = (0,\infty)\ltimes\mathbb{R}$ with the multiplication $(x', y')(x, y) = (x'x, y' + x'y)$, and a left Haar measure $\lambda = dx\,dy/x^2$. A Følner sequence $\{K_n\}_{n=1}^{\infty}$ for G is given by the trapezoidal regions K_n defined by:

$$K_n = \{(x,y) \in \mathbb{R}^2 : 1/n \leqslant x \leqslant n,\ -nx \leqslant y \leqslant nx\}.$$

Now, let $q : G \longrightarrow G/\mathbb{R} \cong (0,\infty)$, be the canonical quotient homomorphism, and $\psi : (0,\infty) \longrightarrow (0,\infty)$ by any continuous homomorphism with dense range. Then $\varphi = \psi\circ q : G \longrightarrow (0,\infty)$ is a continuous homomorphism with dense range in $(0,\infty)$, and therefore φ is a.p.-equidistributed along $\{K_n\}_{n=1}^{\infty}$.

As an example, let $\psi(x) = x^\alpha$, where $\alpha > 0$ is fixed. Then $\varphi(x, y) = \psi\circ q(x, y) = x^\alpha$, and hence by definition, for every $f \in AP((0,\infty))$ we have

$$\langle M, f\rangle = \lim_{n\to\infty}\frac{1}{\lambda(K_n)}\int_{K_n}(f\circ\varphi)(x,y)\,d\lambda = \lim_{n\to\infty}\frac{1}{4n\log n}\int_{1/n}^{n}\int_{-nx}^{nx} f(x^\alpha)\frac{dy\,dx}{x^2} = \lim_{n\to\infty}\frac{1}{2\log n}\int_{1/n}^{n}\frac{f(x^\alpha)}{x}\,dx.$$

Main Result

Theorem 1 (Weyl's criterion): Let S be a locally compact space equipped with a regular Borel measure λ and a monotone compact cover $\{K_\alpha\}_{\alpha\in I}$. Let H be a topological group. Then a continuous mapping $\varphi : S \longrightarrow H$ is a.p.-equidistributed along $\{K_\alpha\}_{\alpha\in I}$ if and only if

$$\lim_{\alpha\in I}\frac{1}{\lambda(K_\alpha)}\int_{K_\alpha}(\sigma_{ij}\circ\varphi)(s)\,d\lambda(s) = 0, \qquad (4)$$

for all nontrivial $\sigma \in \mathscr{C}_H$, $1 \le i, j \le d_\sigma$.

Proof of Theorem 1: First, suppose that φ is a.p.-equidistributed along $\{K_\alpha\}_{\alpha\in I}$, and let M be the almost periodic mean on $AP(H)$. Since $\sigma_{ij} \in AP(H)$ $(1 \le i, j \le d_\sigma)$, by Definition 2 we can write

$$\lim_{\alpha\in I}\frac{1}{\lambda(K_\alpha)}\int_{K_\alpha}(\sigma_{ij}\circ\varphi)(s)\,d\lambda(s) = \langle M, \sigma_{ij}\rangle. \qquad (5)$$

Consider the inner product on $AP(H)$ defined by $(f|g) = M(f\bar{g})$, $f, g \in AP(H)$. By assumption $\sigma \not\cong 1_H$, and therefore we can use the orthogonality relation between coefficient functions of representations in $\mathscr{C}_H$ to write

$$\langle M, \sigma_{ij}\rangle = \langle M, \sigma_{ij}\bar{1}_H\rangle = (\sigma_{ij}|1_H) = 0. \qquad (6)$$

The identities (5) and (6) imply (4).

Conversely, suppose that (4) holds. We need to show that

$$\lim_{\alpha\in I}\frac{1}{\lambda(K_\alpha)}\int_{K_\alpha}(f\circ\varphi)(s)\,d\lambda(s) = \langle M, f\rangle \qquad \text{for all } f \in AP(H). \qquad (7)$$

Since M is a mean on $AP(H)$, we have $\langle M, 1_H\rangle = 1$, and hence both sides of (7) are equal to 1 for $f = 1_H$. If $f = \sigma_{ij}$, $\sigma \in \mathscr{C}_H$, $\sigma \not\cong 1_H$, $1 \le i, j \le d_\sigma$, then (7) holds by (4) and (6). Thus (7) holds for all linear combinations of coefficient functions of representations in $\mathscr{C}_H$. Since such functions are uniformly dense in $AP(H)$, (7) follows by a 3ϵ-argument.

Applications

Theorem 2: Let H be a topological group, $m \geqslant 1$, and $\varphi : \mathbb{N}^m \longrightarrow H$ a function. Suppose that there exists some $1 \leqslant l \leqslant m$, such that for every $h \in \mathbb{N}$, the function

$$\mathbb{N}^m \longrightarrow H, \quad \mathbf{n} \mapsto \varphi(n_1, \ldots, n_l + h, \ldots, n_m)\varphi(n_1, \ldots, n_l, \ldots, n_m)^{-1} \qquad (8)$$

is a.p.-equidistributed along the monotone cover $\{K_\mathbf{N}\}_{\mathbf{N}\in\mathbb{N}^m}$. Then for all $\mathbf{q} \in \mathbb{N}^m$, and all $\mathbf{r} \in \mathbb{Z}^m$ with $r_j \geqslant 0$ $(1 \leqslant j \leqslant m)$, the function

$$\psi : \mathbb{N}^m \longrightarrow H, \quad \mathbf{n} \mapsto \varphi(\mathbf{n}\mathbf{q} + \mathbf{r}) \qquad (9)$$

is a.p.-equidistributed along $\{K_\mathbf{N}\}_{\mathbf{N}\in\mathbb{N}^m}$.

Theorem 3: Let $m \geqslant 1$ and $\{K_N\}_{N\in\mathbb{N}^m}$ be the monotone cover. Let $p(x_1, \ldots, x_m)$ be a polynomial in m variables and with real coefficients that contains at least one nonconstant term with an irrational coefficient. Then for all integers $q_j \geqslant 1$, $r_j \geqslant 0$ $(j = 1, \ldots, m)$, the function

$$\psi : \mathbb{N}^m \longrightarrow \mathbb{T}, \quad (n_1, \ldots, n_m) \mapsto e^{2\pi i p(n_1 q_1 + r_1, \ldots, n_m q_m + r_m)}, \qquad (10)$$

is μ-equidistributed along $\{K_N\}_{N\in\mathbb{N}^m}$, where μ is the normalized Lebesgue measure on $\mathbb{T}$.

Remark: Theorem 2 is a generalization of Hlawka's equidistribution result while Theorem 3 is a generalization of van der Corput's result.

Further Questions

- We believe that our results for equidistribution of continuous functions with values in topological groups can be extended to continuous functions with values in topological semigroups.
- We will try to explore whether above results have any relationship with the ergodic theory, such as the mean ergodic theorem and so on.

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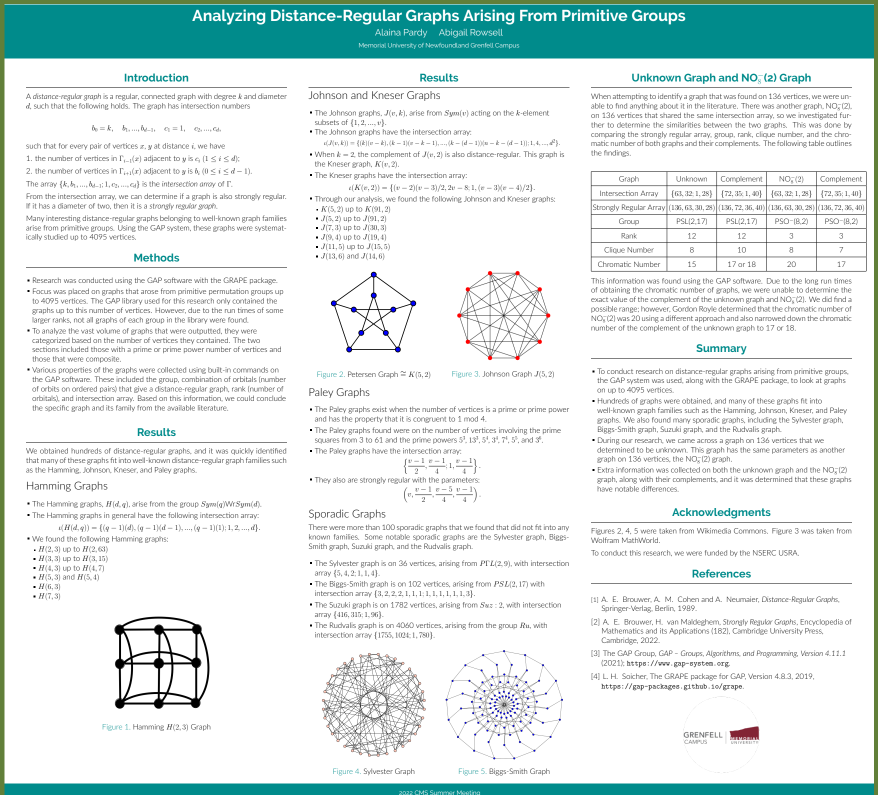
Analyzing Distance-Regular Graphs Arizing From Primitive Groups

- Alaina Pardy and Abigail Rowsell (Memorial University of Newfoundland Grenfell Campus)


Analyzing Distance-Regular Graphs Arising From Primitive Groups

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Introduction

A *distance-regular graph* is a regular, connected graph with degree k and diameter d, such that the following holds. The graph has intersection numbers

$$b_0 = k, \quad b_1, \ldots, b_{d-1}, \quad c_1 = 1, \quad c_2, \ldots, c_d,$$

such that for every pair of vertices x, y at distance i, we have

1. the number of vertices in $\Gamma_{i-1}(x)$ adjacent to y is c_i $(1 \le i \le d)$;
2. the number of vertices in $\Gamma_{i+1}(x)$ adjacent to y is b_i $(0 \le i \le d-1)$.

The array $\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$ is the *intersection array* of Γ.

From the intersection array, we can determine if a graph is also strongly regular. If it has a diameter of two, then it is a strongly regular graph.

Many interesting distance-regular graphs belonging to well-known graph families arise from primitive groups. Using the GAP system, these graphs were systematically studied up to 4095 vertices.

Methods

- Research was conducted using the GAP software with the GRAPE package.
- Focus was placed on graphs that arose from primitive permutation groups up to 4095 vertices. The GAP library used for this research only contained the graphs up to this number of vertices. However, due to the run times of some larger ranks, not all graphs of each group in the library were found.
- To analyze the vast volume of graphs that were outputted, they were categorized based on the number of vertices they contained. The two sections included those with a prime or prime power number of vertices and those that were composite.
- Various properties of the graphs were collected using built-in commands on the GAP software. These included the group, combination of orbitals (number of orbits on ordered pairs) that give a distance-regular graph, rank (number of orbitals), and intersection array. Based on this information, we could conclude the specific graph and its family from the available literature.

Results

We obtained hundreds of distance-regular graphs, and it was quickly identified that many of these graphs fit into well-known distance-regular graph families such as the Hamming, Johnson, Kneser, and Paley graphs.

Hamming Graphs

- The Hamming graphs, $H(d, q)$, arise from the group $Sym(q) \wr Sym(d)$.
- The Hamming graphs in general have the following intersection array: $\iota(H(d,q)) = \{(q-1)(d), (q-1)(d-1), \ldots, (q-1)(1); 1, 2, \ldots, d\}$.
- We found the following Hamming graphs:
  - $H(2,3)$ up to $H(2,63)$
  - $H(3,3)$ up to $H(3,15)$
  - $H(4,3)$ up to $H(4,7)$
  - $H(5,3)$ and $H(5,4)$
  - $H(6,3)$
  - $H(7,3)$

Figure 1. Hamming $H(2,3)$ Graph

Results

Johnson and Kneser Graphs

- The Johnson graphs, $J(v, k)$, arise from $Sym(v)$ acting on the k-element subsets of $\{1, 2, \ldots, v\}$.
- The Johnson graphs have the intersection array: $\iota(J(v,k)) = \{(k)(v-k), (k-1)(v-k-1), \ldots, (k-(d-1))(v-k-(d-1)); 1, 4, \ldots, d^2\}$.
- When $k = 2$, the complement of $J(v, 2)$ is also distance-regular. This graph is the Kneser graph, $K(v, 2)$.
- The Kneser graphs have the intersection array: $\iota(K(v,2)) = \{(v-2)(v-3)/2, 2v-8; 1, (v-3)(v-4)/2\}$.
- Through our analysis, we found the following Johnson and Kneser graphs:
  - $K(5,2)$ up to $K(91,2)$
  - $J(5,2)$ up to $J(91,2)$
  - $J(7,3)$ up to $J(30,3)$
  - $J(9,4)$ up to $J(19,4)$
  - $J(11,5)$ up to $J(15,5)$
  - $J(13,6)$ and $J(14,6)$

Figure 2. Petersen Graph $\cong K(5,2)$ Figure 3. Johnson Graph $J(5,2)$

Paley Graphs

- The Paley graphs exist when the number of vertices is a prime or prime power and has the property that it is congruent to 1 mod 4.
- The Paley graphs found were on the number of vertices involving the prime squares from 3 to 61 and the prime powers $3^3, 13^3, 5^3, 3^4, 7^4, 5^5$, and 3^6.
- The Paley graphs have the intersection array: $\left\{\frac{v-1}{2}, \frac{v-1}{4}; 1, \frac{v-1}{4}\right\}$.
- They also are strongly regular with the parameters: $\left(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4}\right)$.

Sporadic Graphs

There were more than 100 sporadic graphs that we found that did not fit into any known families. Some notable sporadic graphs are the Sylvester graph, Biggs-Smith graph, Suzuki graph, and the Rudvalis graph.

- The Sylvester graph is on 36 vertices, arising from $P\Gamma L(2, 9)$, with intersection array $\{5, 4, 2; 1, 1, 4\}$.
- The Biggs-Smith graph is on 102 vertices, arising from $PSL(2, 17)$ with intersection array $\{3, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 1, 1, 3\}$.
- The Suzuki graph is on 1782 vertices, arising from $Suz : 2$, with intersection array $\{416, 315; 1, 96\}$.
- The Rudvalis graph is on 4060 vertices, arising from the group Ru, with intersection array $\{1755, 1024; 1, 780\}$.

Figure 4. Sylvester Graph Figure 5. Biggs-Smith Graph

Unknown Graph and $NO_8^-(2)$ Graph

When attempting to identify a graph that was found on 136 vertices, we were unable to find anything about it in the literature. There was another graph, $NO_8^-(2)$, on 136 vertices that shared the same intersection array, so we investigated further to determine the similarities between the two graphs. This was done by comparing the strongly regular array, group, rank, clique number, and the chromatic number of both graphs and their complements. The following table outlines the findings.

Graph	Unknown	Complement	$NO_8^-(2)$	Complement
Intersection Array	$\{63, 32; 1, 28\}$	$\{72, 35; 1, 40\}$	$\{63, 32; 1, 28\}$	$\{72, 35; 1, 40\}$
Strongly Regular Array	$(136, 63, 30, 28)$	$(136, 72, 36, 40)$	$(136, 63, 30, 28)$	$(136, 72, 36, 40)$
Group	PSL(2,17)	PSL(2,17)	$PSO^-(8,2)$	$PSO^-(8,2)$
Rank	12	12	3	3
Clique Number	8	10	8	7
Chromatic Number	15	17 or 18	20	17

This information was found using the GAP software. Due to the long run times of obtaining the chromatic number of graphs, we were unable to determine the exact value of the complement of the unknown graph and $NO_8^-(2)$. We did find a possible range; however, Gordon Royle determined that the chromatic number of $NO_8^-(2)$ was 20 using a different approach and also narrowed down the chromatic number of the complement of the unknown graph to 17 or 18.

Summary

- To conduct research on distance-regular graphs arising from primitive groups, the GAP system was used, along with the GRAPE package, to look at graphs on up to 4095 vertices.
- Hundreds of graphs were obtained, and many of these graphs fit into well-known graph families such as the Hamming, Johnson, Kneser, and Paley graphs. We also found many sporadic graphs, including the Sylvester graph, Biggs-Smith graph, Suzuki graph, and the Rudvalis graph.
- During our research, we came across a graph on 136 vertices that we determined to be unknown. This graph has the same parameters as another graph on 136 vertices, the $NO_8^-(2)$ graph.
- Extra information was collected on both the unknown graph and the $NO_8^-(2)$ graph, along with their complements, and it was determined that these graphs have notable differences.

Acknowledgements

Figures 2, 4, 5 were taken from Wikimedia Commons. Figure 3 was taken from Wolfram MathWorld.

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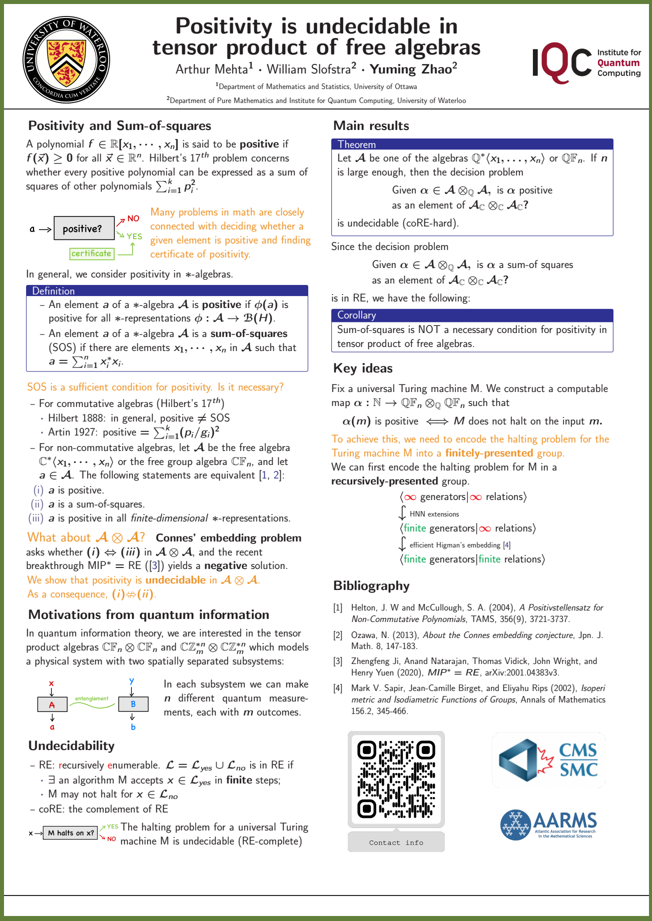
- Yuming Zhao (University of Waterloo)



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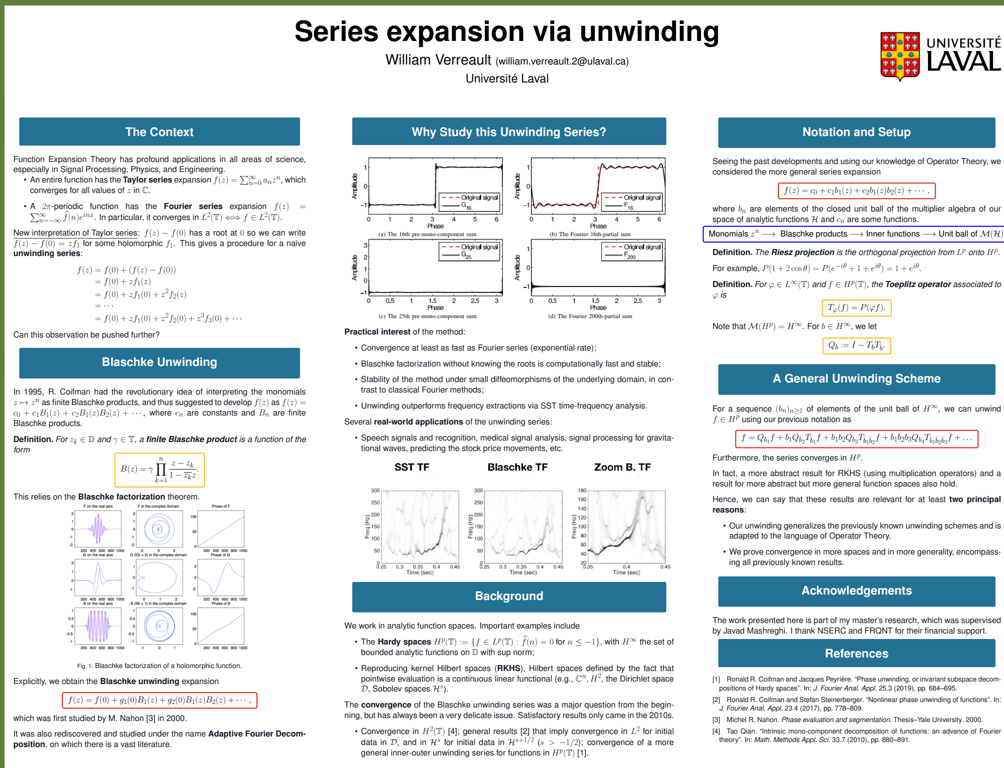

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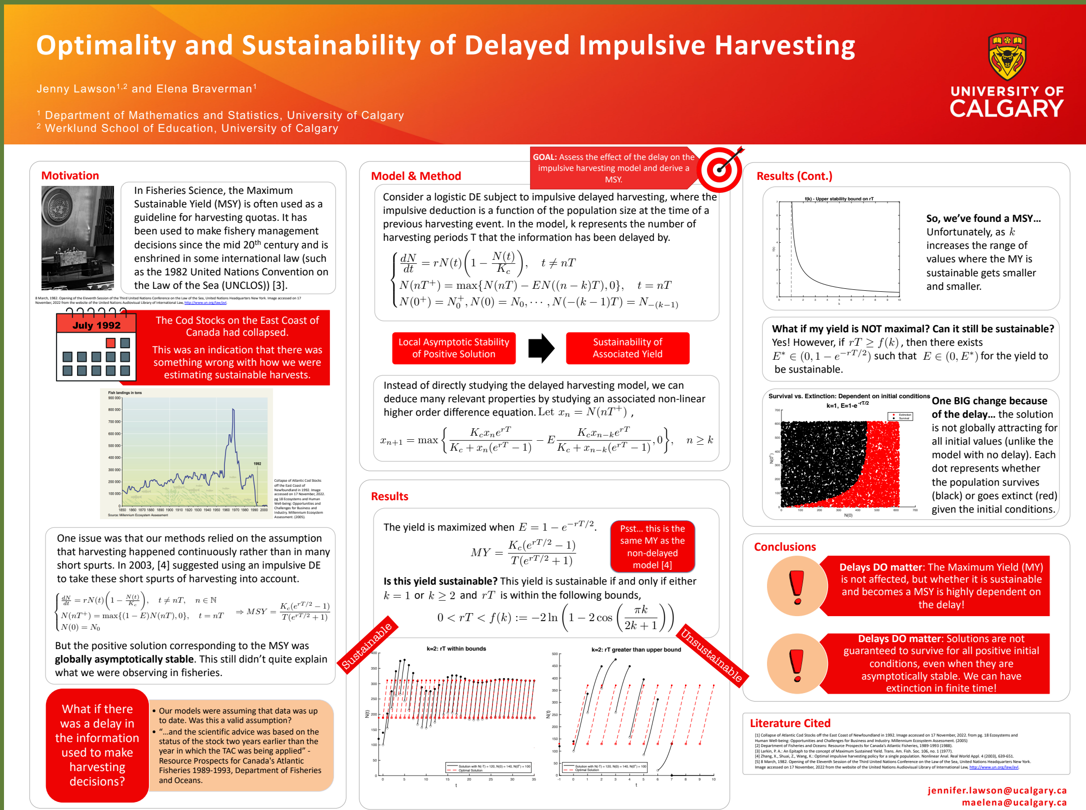

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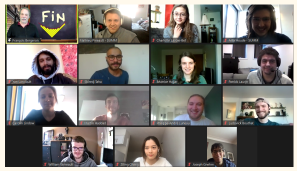

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